“Calibeating”: Beating Forecasters at Their Own Game

Sergiu Hart

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Joint work with

Dean P. Foster

University of Pennsylvania &

Amazon Research NY

Papers

Papers

- Sergiu Hart
  “Calibration: The Minimax Proof”, 1995 [2021]
  www.ma.huji.ac.il/hart/publ.html#calib-minmax

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- Sergiu Hart
  "Calibration: The Minimax Proof", 1995 [2021]
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- Dean P. Foster and Sergiu Hart
  "Smooth Calibration, Leaky Forecasts, Finite Recall, and Nash Dynamics"
  Games and Economic Behavior 2018
  www.ma.huji.ac.il/hart/publ.html#calib-eq

Papers

Papers

- Dean P. Foster and Sergiu Hart
  “Forecast Hedging and Calibration”
  Journal of Political Economy 2021

  www.ma.huji.ac.il/hart/publ.html#calib-int

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- Dean P. Foster and Sergiu Hart
  " 'Calibeating': Beating Forecasters at Their Own Game"
  Theoretical Economics 2023

  www.ma.huji.ac.il/hart/publ.html#calib-beat

Calibration

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- Forecaster says: "***The probability of rain tomorrow is*** p"

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  - for every forecast p:
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Calibration

- Forecaster says: "***The probability of rain tomorrow is** p*"
- Forecaster is **CALIBRATED** if
  - for every forecast p:
    in the days when the forecast was p, the proportion of rainy days equals p
    (or: is close to p in the long run)

Calibration

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CALIBRATION ***can be guaranteed***

(no matter what the weather will be)

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- Foster 1999: simple procedure

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- Foster 1999: simple procedure
- Foster and Hart 2016 [publ 2021]: simplest procedure, by "Forecast Hedging"

Forecast-Hedging (FH)

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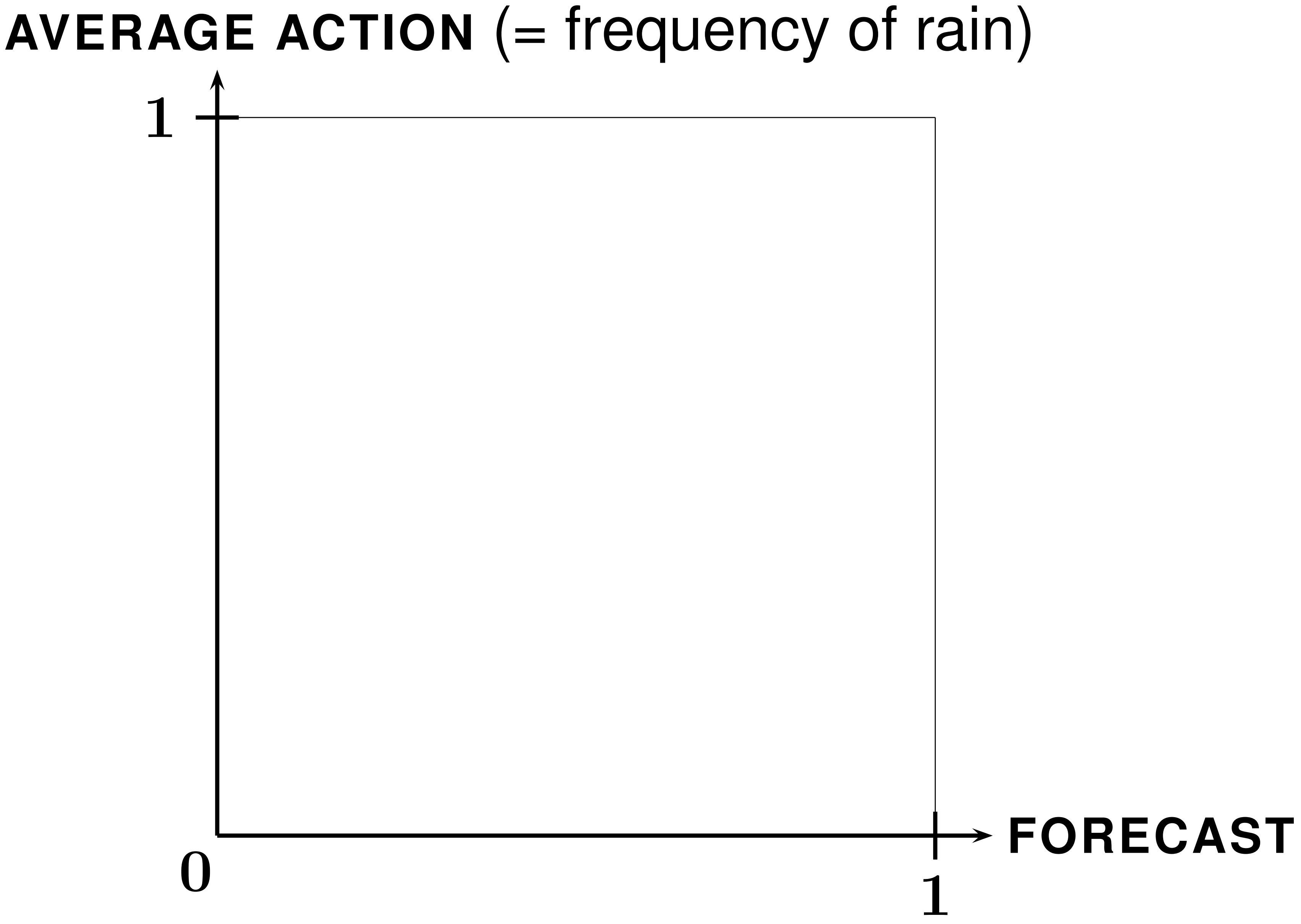

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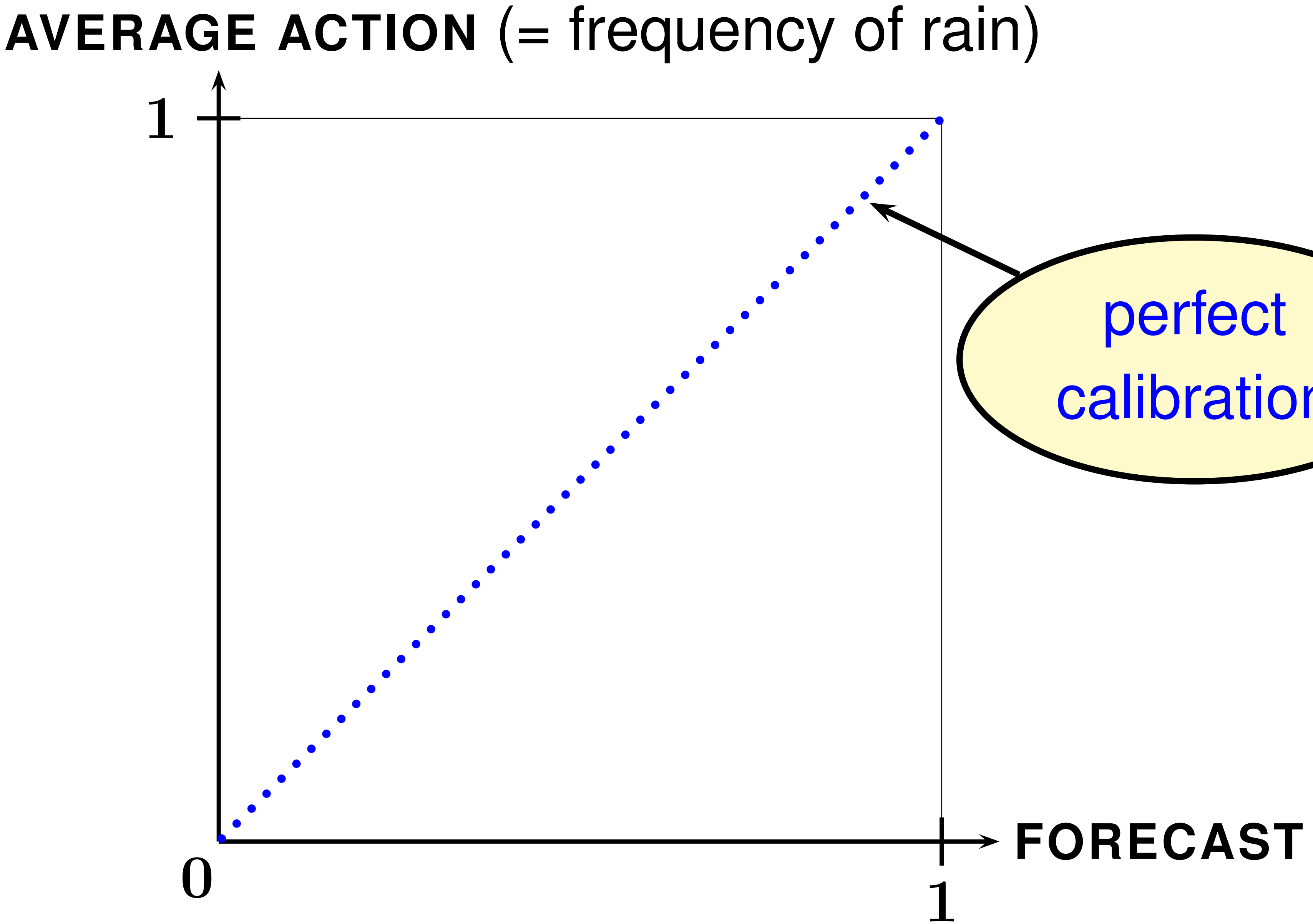

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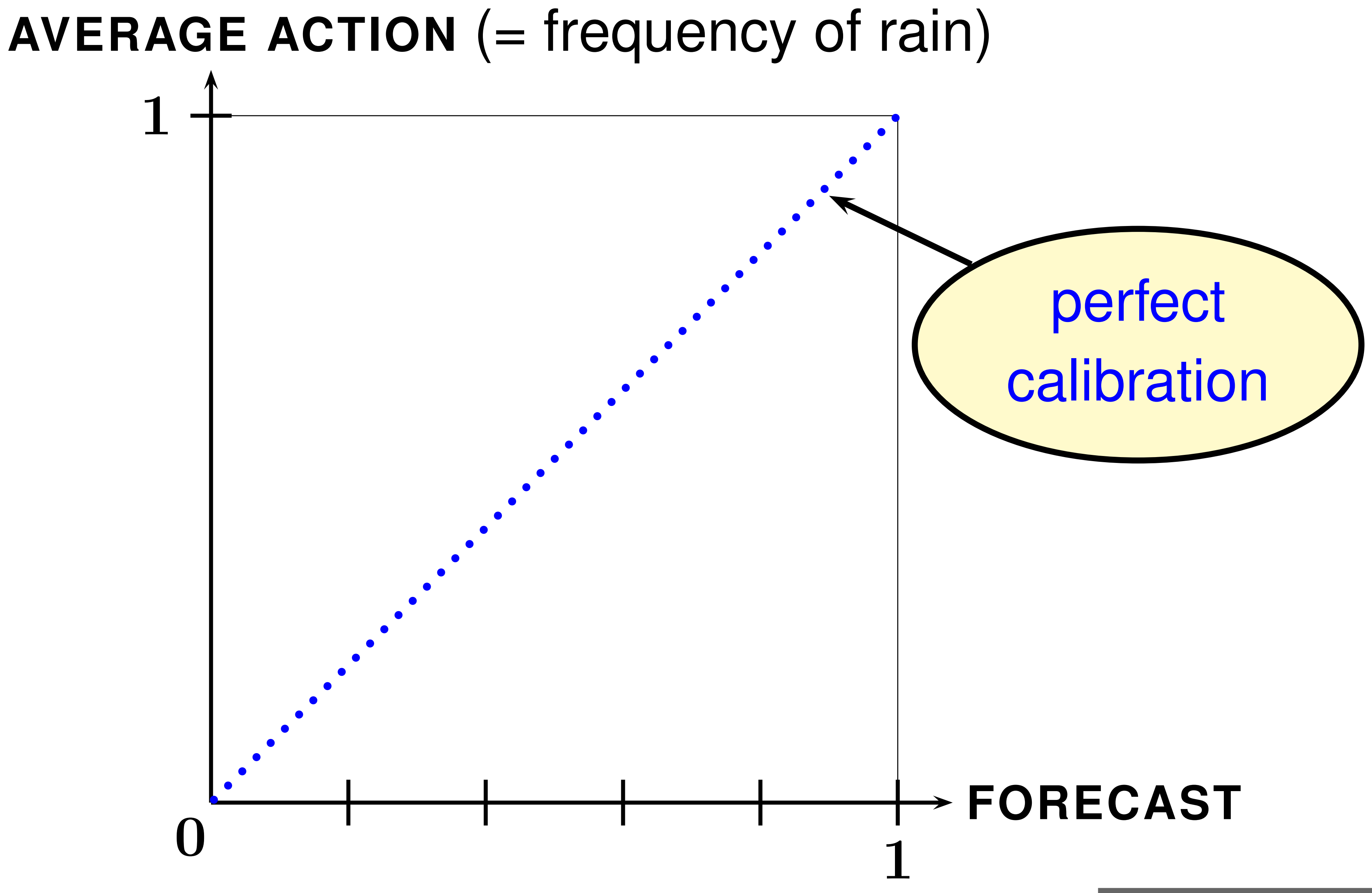

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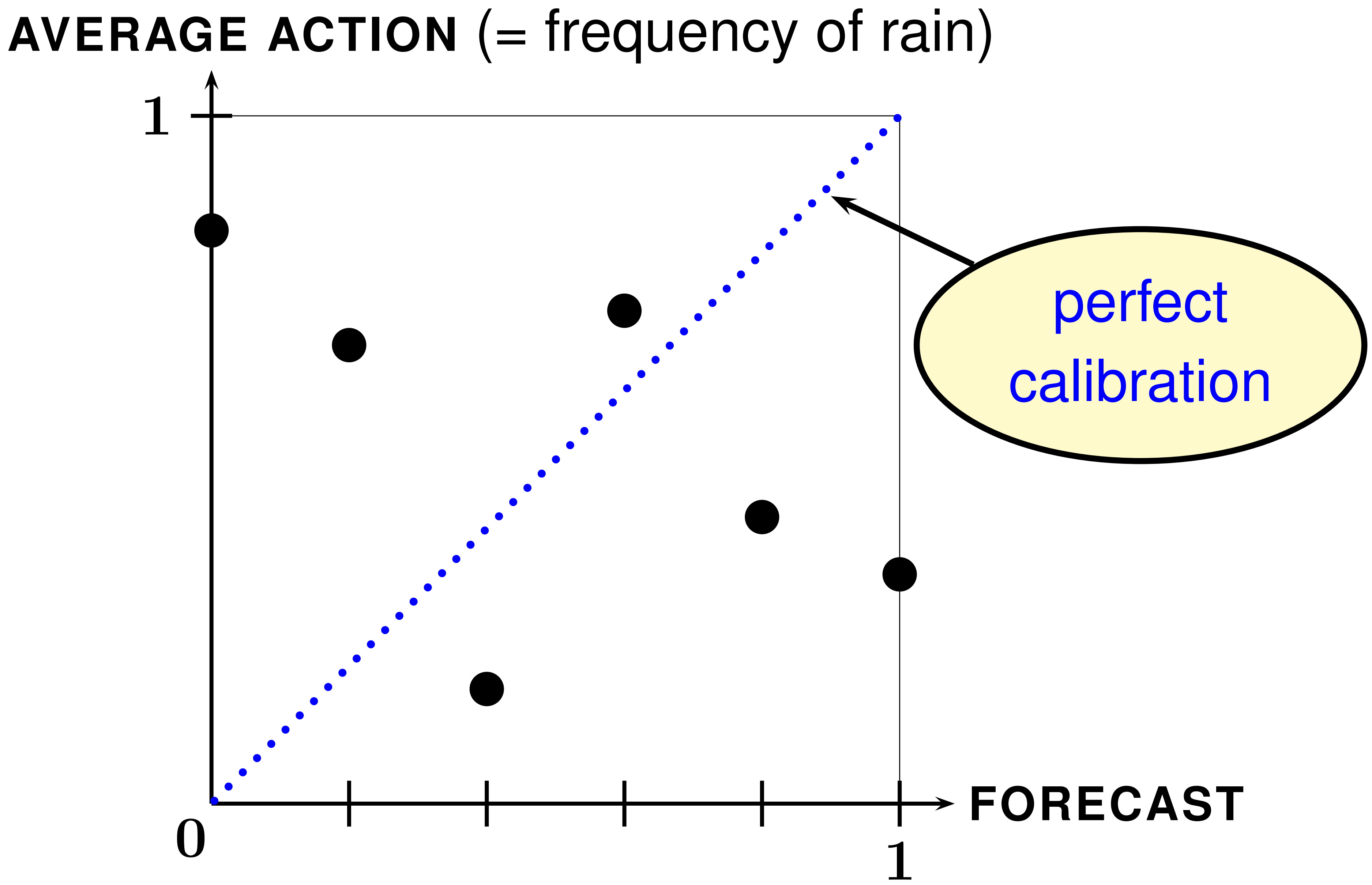

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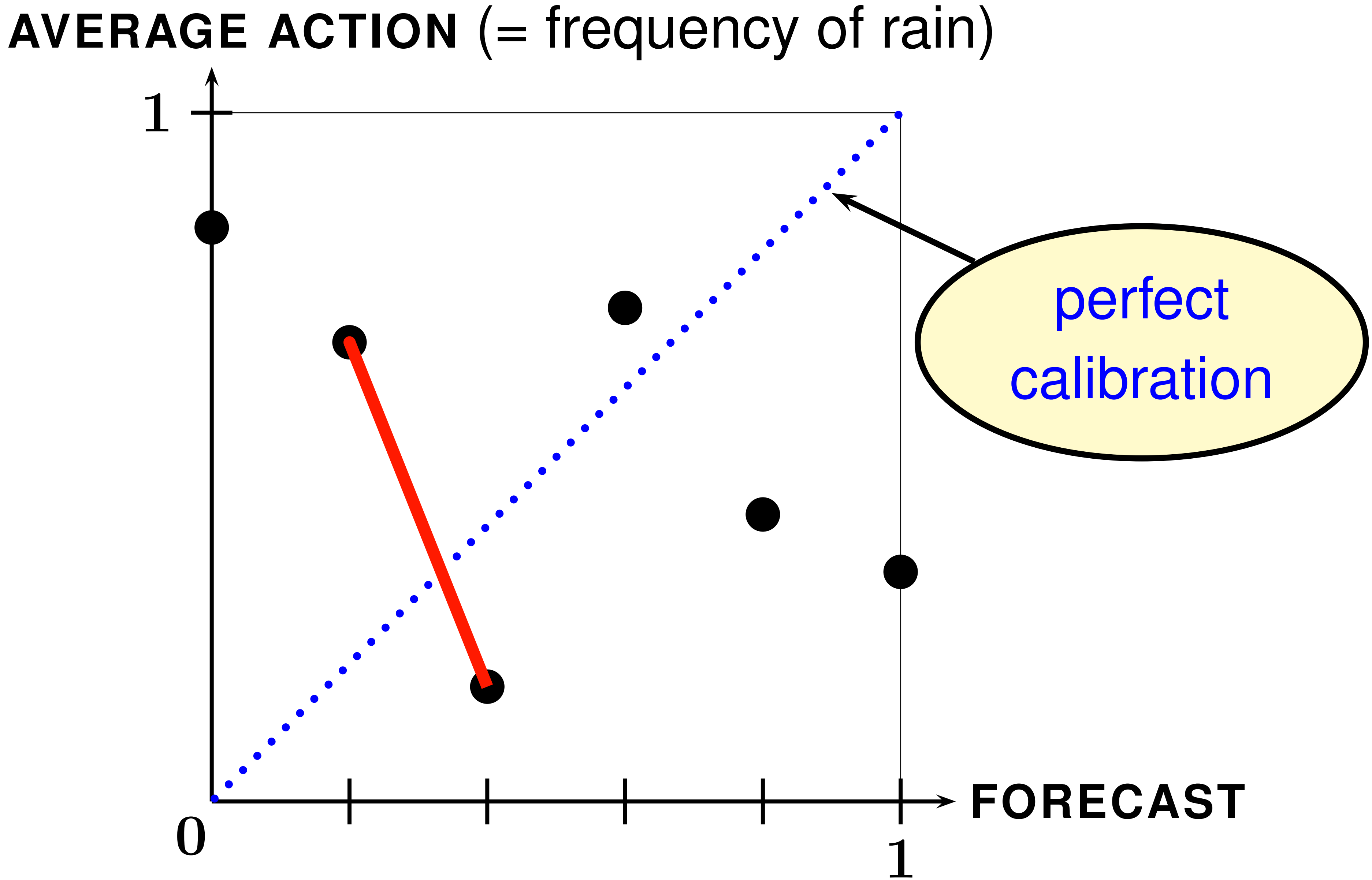

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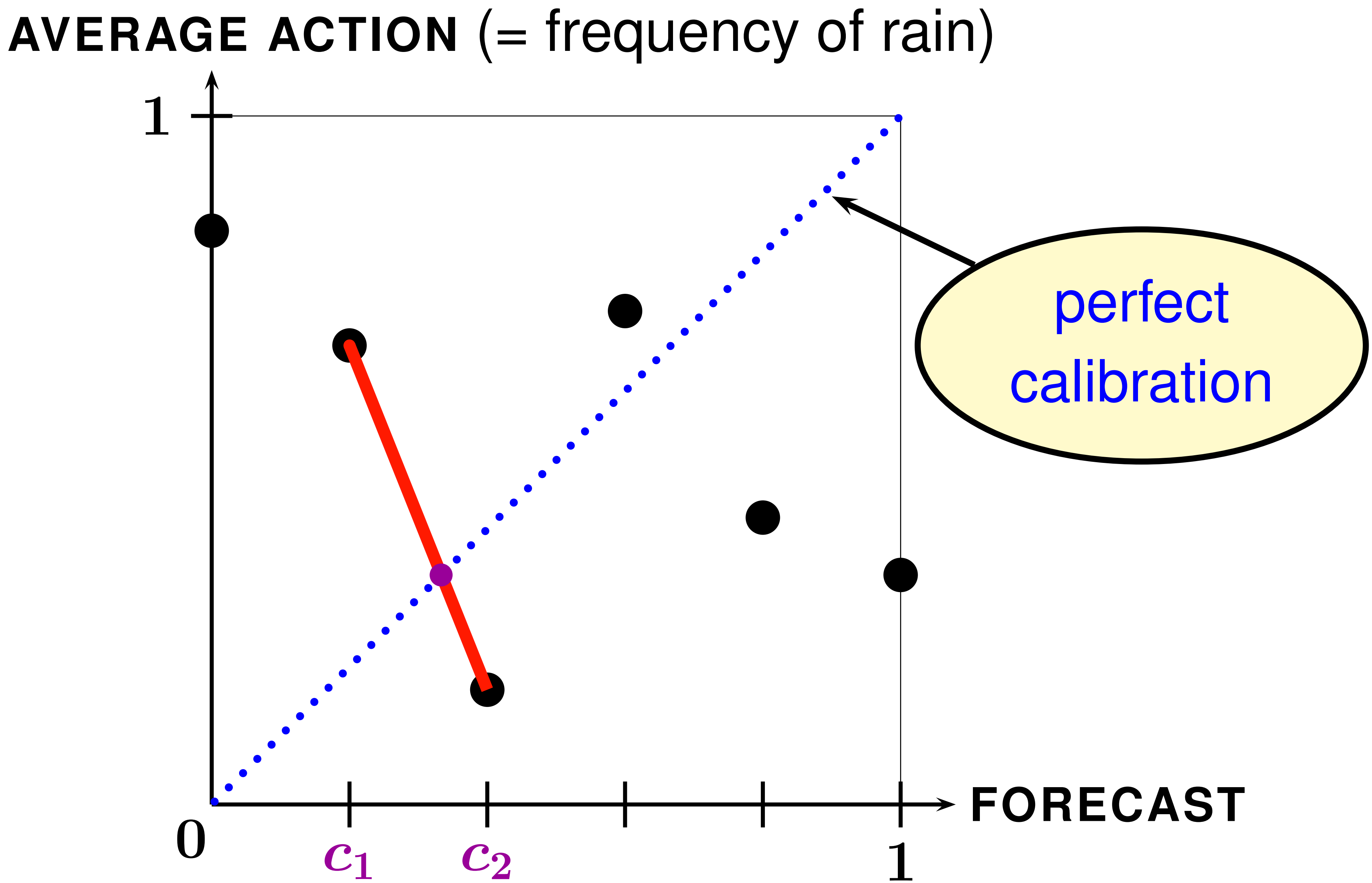

Calibration in Practice

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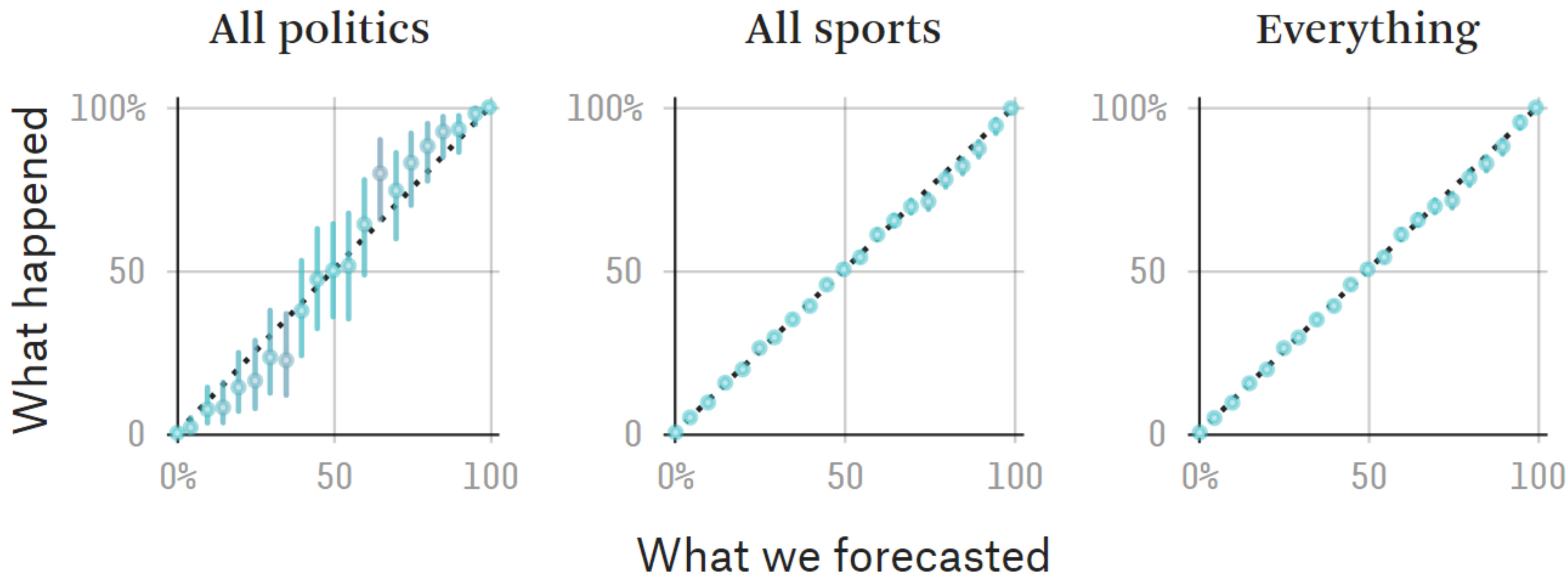
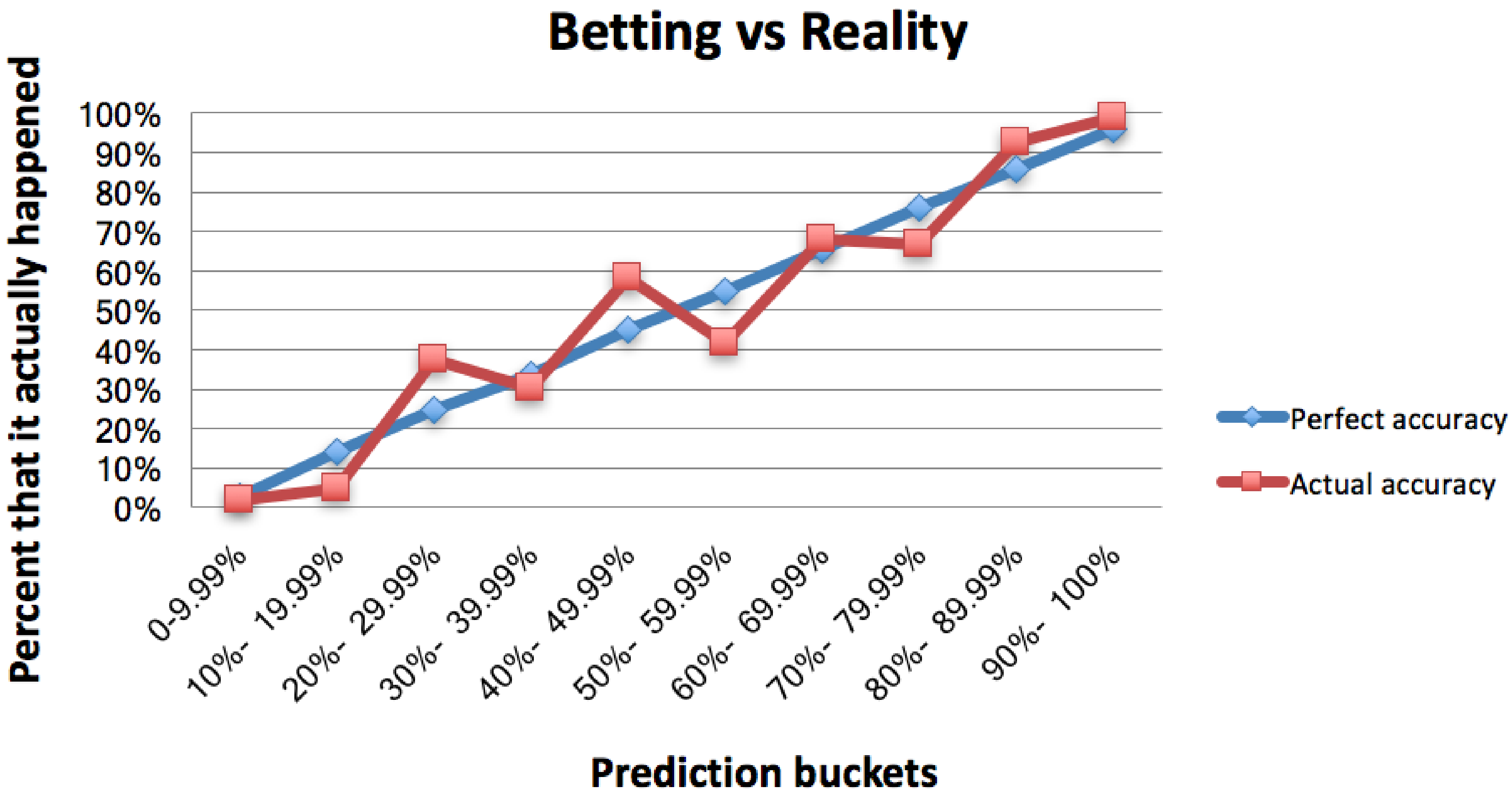


Calibration plots of FiveThirtyEight.com
(as of June 2019)

Calibration in Practice



Calibration plot of ElectionBettingOdds.com (2016 – 2018)

Example

Example

time	1	2	3	4	5	6	...

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time	**1**	**2**	**3**	**4**	**5**	**6**	...
rain	1	0	1	0	1	0	

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
F1	100%	0%	100%	0%	100%	0%	

Example

time	**1**	**2**	**3**	**4**	**5**	**6**	...
rain	1	0	1	0	1	0	
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F2	50%	50%	50%	50%	50%	50%	

Example

time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
F1	100%	0%	100%	0%	100%	0%	
F2	50%	50%	50%	50%	50%	50%	

F1: **CALIBRATION** $= \mathbf{0}$

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F1: **CALIBRATION** $= \mathbf{0}$

F2: **CALIBRATION** $= \mathbf{0}$

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F1: **CALIBRATION** $= 0$ **IN-BIN VARIANCE** $= 0$

F2: **CALIBRATION** $= 0$

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
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F1: **CALIBRATION** $= \mathbf{0}$ **IN-BIN VARIANCE** $= \mathbf{0}$

F2: **CALIBRATION** $= \mathbf{0}$ **IN-BIN VARIANCE** $= \frac{\mathbf{1}}{\mathbf{4}}$

Notations

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$$\bar{a}(x) = \frac{\sum_{t=1}^{T} \mathbf{1}_x(c_t)\, a_t}{\sum_{t=1}^{T} \mathbf{1}_x(c_t)}$$

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Proof.

$$\mathbb{E}[(X - c)^2] = \mathbb{V}ar(X) + (\bar{X} - c)^2$$

where c is a constant and X is a random variable with $\bar{X} = \mathbb{E}[X]$

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F1: $\mathcal{K} = 0 \quad \mathcal{R} = 0$

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F1: $\mathcal{K} = \mathbf{0}$ $\mathcal{R} = \mathbf{0}$ $\mathcal{B} = \mathbf{0}$

F2: $\mathcal{K} = \mathbf{0}$ $\mathcal{R} = \frac{\mathbf{1}}{\mathbf{4}}$ $\mathcal{B} = \frac{\mathbf{1}}{\mathbf{4}}$

“Experts”

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Testing experts:

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Testing experts:

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✗ CALIBRATION score

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- Recognize ***patterns*** and ***regularities*** in the data

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$\Leftrightarrow$ ***LOW*** **REFINEMENT SCORE**

“Expertise” and Calibration

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Can one **GAIN CALIBRATION**

without **LOSING “EXPERTISE”** ?

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- Can one get $\mathcal{K}$ to 0 without increasing $\mathcal{R}$?

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<u>Question:</u>

Can one **GAIN CALIBRATION**
<u>without</u> **LOSING "EXPERTISE"** ?

- Can one get $\mathcal{K}$ to 0 without increasing $\mathcal{R}$?
- Can one decrease $\mathcal{B} = \mathcal{R} + \mathcal{K}$ by $\mathcal{K}$?

“Expertise” and Calibration

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Question:

Can one do this **ONLINE** ?

- Consider a forecasting sequence b_t (in a [finite] set B)

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- At each time $\boldsymbol{t}$ generate a forecast $\boldsymbol{c}_t$

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$$\mathcal{B}_T^{\mathrm{c}} \leq \mathcal{B}_T^{\mathrm{b}} - \mathcal{K}_T^{\mathrm{b}} + \mathrm{o}(1) \quad \text{as } T \to \infty$$

for ALL sequences a_t and b_t (uniformly)

- Consider a forecasting sequence b_t (in a [finite] set B)
- At each time t generate a forecast c_t
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"Calibeating"

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c "**BEATS**" b by b's **CALIBRATION** score

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$\boxed{c \text{ “BEATS” } b \text{ by } b\text{'s CALIBRATION score}}$

- GUARANTEED for ALL sequences of actions and forecasts

Example

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
rain	1	0	1	0	1	0	
$\boldsymbol{b}$	80%	40%	80%	40%	80%	40%	

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$\boldsymbol{b}$: $\mathcal{K}^{\mathrm{b}} = \mathbf{0.1}$ $\quad \mathcal{R}^{\mathrm{b}} = \mathbf{0}$ $\quad \mathcal{B}^{\mathrm{b}} = \mathbf{0.1}$

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$\boldsymbol{c}$	100%	0%	100%	0%	100%	0%	

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time	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	...
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$\boldsymbol{c}$	100%	0%	100%	0%	100%	0%	

$\boldsymbol{b}$: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\mathcal{R}^{\mathrm{b}} = 0$ $\mathcal{B}^{\mathrm{b}} = 0.1$

$\boldsymbol{c}$: $\mathcal{K}^{\mathrm{c}} = 0$ $\mathcal{R}^{\mathrm{c}} = 0$ $\mathcal{B}^{\mathrm{c}} = 0$

Calibeating

time	1	2	3	4	5	6	...
rain	1	0	1	0	1	0	
b	80%	40%	80%	40%	80%	40%	
c	100%	0%	100%	0%	100%	0%	

b: $\mathcal{K}^{\mathrm{b}} = 0.1$ $\mathcal{R}^{\mathrm{b}} = 0$ $\mathcal{B}^{\mathrm{b}} = 0.1$

c: $\mathcal{K}^{\mathrm{c}} = 0$ $\mathcal{R}^{\mathrm{c}} = 0$ $\mathcal{B}^{\mathrm{c}} = 0$

c ***calibeats*** b: $\mathcal{B}^{\mathrm{c}} \leq \mathcal{B}^{\mathrm{b}} - \mathcal{K}^{\mathrm{b}}$

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$\boldsymbol{c}$	100%	0%	100%	0%	100%	0%	

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$\boldsymbol{c}$: $\mathcal{K}^{\mathrm{c}} = 0$ $\quad \mathcal{R}^{\mathrm{c}} = 0$ $\quad \mathcal{B}^{\mathrm{c}} = 0$

c ***calibeats*** b: $\mathcal{B}^{\mathrm{c}} \leq \mathcal{B}^{\mathrm{b}} - \mathcal{K}^{\mathrm{b}} = \mathcal{R}^{\mathrm{b}}$

Calibeating

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(that was easy ...)

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Can one* CALIBEAT *in general, non-stationary, situations ?

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- **Bin averages** do not converge
- ONLINE
- GUARANTEED (even against adversary)

Calibeating

Calibeating

Theorem

There exists a **CALIBEATING** procedure

A Way to Calibeat

A Way to Calibeat

Theorem

The procedure

$$\boxed{c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)}$$

GUARANTEES b-CALIBEATING

A Simple Way to Calibeat

Theorem

The procedure

$$\boxed{c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)}$$

GUARANTEES b-CALIBEATING

Forecast the average action of the current b-forecast

Proof

Proof

$$\mathbb{V}\mathbf{ar} = \frac{\mathbf{1}}{\boldsymbol{T}} \sum_{\boldsymbol{t}=\mathbf{1}}^{\boldsymbol{T}} \|\boldsymbol{x}_{\boldsymbol{t}} - \bar{\boldsymbol{x}}_{\boldsymbol{T}}\|^{\mathbf{2}}$$

Proof

$$\begin{aligned}
\mathbb{V}\mathrm{ar} &= \frac{1}{T}\sum_{t=1}^{T}\left\|x_t - \bar{x}_T\right\|^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}\left(1-\frac{1}{t}\right)\left\|x_t - \bar{x}_{t-1}\right\|^2
\end{aligned}$$

Proof

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\end{aligned}$$

(*) $\mathrm{o}(1) = \mathrm{O}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{1}{t}\right) = \mathrm{O}\left(\frac{\log T}{T}\right)$

Proof

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Proof: "Online Variance"

$$
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&= \underbrace{\frac{1}{T}\sum_{t=1}^{T}\|x_t - \bar{x}_{t-1}\|^2}_{} - \mathrm{o}(1) \\
&= \widetilde{\mathbb{V}\mathrm{ar}} - \mathrm{o}(1)
\end{aligned}
$$

Proof: "Online Variance"

$$\mathbb{V}\mathbf{ar} \;=\; \widetilde{\mathbb{V}\mathbf{ar}} - \mathbf{o}(1)$$

Proof: “Online Refinement”

$$
\begin{aligned}
\mathbb{V}\mathbf{ar} &= \widetilde{\mathbb{V}\mathbf{ar}} - \mathrm{o}(1) \\
\mathcal{R}^{\mathrm{b}} &= \widetilde{\mathcal{R}}^{\mathrm{b}} - \mathrm{o}(1)
\end{aligned}
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&= \frac{1}{T}\sum_{t=1}^{T} \|a_t - \bar{a}_{t-1}(b_t)\|^2 - \mathrm{o}(1)
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Proof: "Online Refinement"

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&= \underbrace{\phantom{\frac{1}{T}\sum_{t=1}^{T} \|a_t - \bar{a}_{t-1}(b_t)\|^2}}_{\mathcal{B}^{\mathrm{c}}} - \mathrm{o}(1)
\end{aligned}$$

$c_t = \bar{a}_{t-1}(b_t)$

Calibeating

Calibeating

Theorem

$$\boxed{c_t = \bar{a}^{\mathrm{b}}_{t-1}(b_t)}$$

GUARANTEES b-CALIBEATING:

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Self-Calibeating

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Self-Calibeating = Calibrating

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“Fixed Point”

How do we get $\boldsymbol{c}_t$ “close to” $\bar{\boldsymbol{a}}_{t-1}(\boldsymbol{c}_t)$?

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- wlog (project $g(x)$ to C) $\quad \boldsymbol{g}:\boldsymbol{D}\longrightarrow \boldsymbol{C}$

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Wrong Proof !

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Right Proof. Use the **MINIMAX** theorem

- $h(x,v) := \|v-x\|^2 - \|v-g(x)\|^2$

Stochastic "Fixed Point"

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Right Proof. Use the MINIMAX theorem

- $h(x, v) := \|v - x\|^2 - \|v - g(x)\|^2$
- For every distribution Q on $v \in C$: taking $x \approx \mathbb{E}_Q[v]$ in D yields $h(x, Q) \le \delta^2$

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- **Obtained by solving a MINIMAX problem (LP)**
- Moreover: solving a **FIXED POINT** problem yields a probability distribution P that is **ALMOST DETERMINISTIC**: its support is included in a ball of size δ

Calibrating

Calibrating

Theorem

There is a stochastic procedure that **GUARANTEES CALIBRATION**

Calibrating

Theorem

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Proof. Self-calibeating + Stochastic Fixed Point

Calibrating

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Note. δ-CALIBRATION

Calibrated Calibeating

Calibrated Calibeating

Theorem

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Calibrated Calibeating

Theorem

There is a stochastic procedure that **GUARANTEES** **CALIBEATING** and **CALIBRATION**

Calibrated Calibeating

Theorem

There is a stochastic procedure that **GUARANTEES** **CALIBEATING** and **CALIBRATION**

Proof. Calibeat the **joint** binning of $\boldsymbol{b}$ and $\boldsymbol{c}$, by applying Stochastic Fixed Point

Continuous Calibration

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- **CONTINUOUS CALIBRATION**: combine the days when the forecast was ***close to*** p

Continuous Calibration

- **CONTINUOUS CALIBRATION**: combine the days when the forecast was ***close to*** p (smooth out the calibration score)

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- **Theorem**:

> There exists a ***deterministic*** procedure that is **CONTINUOUSLY CALIBRATED**.

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Deterministic $\Rightarrow$ holds also when the forecasts are ***"leaked"***

Continuous Calibration

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Foster and Kakade (2004, 2006)
Foster and Hart (2018, **2021**)

Continuous-Calibrated Calibeating

Continuous-Calibrated Calibeating

Theorem

There is a ***deterministic*** procedure that **GUARANTEES CALIBEATING**

Continuous-Calibrated Calibeating

Theorem

There is a ***deterministic*** procedure that **GUARANTEES** **CALIBEATING** and **CONTINUOUS CALIBRATION**

Continuous-Calibrated Calibeating

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There is a ***deterministic*** procedure
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Proof. Calibeat the **joint** binning of $\boldsymbol{b}$ and $\boldsymbol{c}$,
by a Fixed Point result (Brouwer)

Multi-Calibeating

Multi-Calibeating

Theorem

There is a ***deterministic*** procedure
that **GUARANTEES**

simultaneous CALIBEATING
of several forecasters

Multi-Calibeating

Theorem

There is a ***stochastic*** procedure
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simultaneous CALIBEATING
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and **CALIBRATION**

Multi-Calibeating

Theorem

There is a ***stochastic*** procedure
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Proof. Calibeat the **joint** binning

In all the results above:

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	CALIBRATION	
Obtained by	***Minimax***	
Procedure	***stochastic***	

... and Continuous Calibration

In all the results above:

	CALIBRATION	CONTINUOUS CALIBRATION
Obtained by	*Minimax*	*Fixed Point*
Procedure	*stochastic*	*deterministic*

Refinement Score and Brier Score

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Claim. The REFINEMENT score is the ***minimal*** BRIER score over all ***relabelings of the bins***:

Refinement Score and Brier Score

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$$\mathcal{R}_T(\mathrm{b}) = \min_{\phi} \mathcal{B}_T(\phi[\mathrm{b}])$$

Refinement Score and Brier Score

Claim. The REFINEMENT score is the ***minimal*** BRIER score over all ***relabelings of the bins***:

$$\mathcal{R}_T(\mathrm{b}) = \min_{\phi} \mathcal{B}_T(\phi[\mathrm{b}])$$

where the minimum is taken over all

$$\phi : B \longrightarrow \Delta(A)$$

and

$$\phi[\mathrm{b}] \equiv (\phi(b_1), \ldots, \phi(b_T))$$

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Proof. Label each bin b with $\phi(b) = \bar{a}_T(b)$

Calibeating and Brier Score

Calibeating and Brier Score

- c **CALIBEATS** b:

$$\mathcal{B}_T(\mathrm{c}) \leq \mathcal{R}_T(\mathrm{b}) + o(1)$$

Calibeating and Brier Score

- $\mathbf{c}$ **CALIBEATS** b:

$$\mathcal{B}(\mathbf{c}) \leq \mathcal{R}(\mathrm{b})$$

Calibeating and Brier Score

- c CALIBEATS b:

$$\mathcal{B}(c) \leq \mathcal{R}(b)$$

$$\mathcal{B}(c) \leq \min_{\phi} \mathcal{B}(\phi[b])$$

Calibeating and Brier Score

- c CALIBEATS b:

$$\mathcal{B}(\mathrm{c}) \le \mathcal{R}(\mathrm{b})$$

$$\mathcal{B}(\mathrm{c}) \le \min_{\phi} \mathcal{B}(\phi[\mathrm{b}])$$

- c MULTI-CALIBEATS $\mathrm{b}_1, ..., \mathrm{b}_N$:

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- c MULTI-CALIBEATS $b_1, ..., b_N$:

$$\mathcal{B}(c) \leq \min_{\phi} \mathcal{B}(\phi[b_1, ..., b_N])$$

where the minimum is taken over all relabelings of the joint bins

$$\phi : B_1 \times ... \times B_N \to \Delta(A)$$

Calibeating and Brier Score

- c **MULTI-CALIBEATS** $\mathrm{b}_1, ..., \mathrm{b}_N$:

$$\mathcal{B}(\mathrm{c}) \leq \min_{\phi} \mathcal{B}(\phi[\mathrm{b}_1, ..., \mathrm{b}_N])$$

Calibeating and Experts

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Calibeating and Experts

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$$\mathcal{B}(c) \le \min_{\phi} \mathcal{B}(\phi[b_1, \ldots, b_N])$$

- c is **AS STRONG AS** $b_1, \ldots, b_N$:

$$\mathcal{B}(c) \le \min_{1 \le n \le N} \mathcal{B}(b_n)$$

Calibeating and Experts

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$$\mathcal{B}(\mathbf{c}) \leq \min_{1 \leq n \leq N} \mathcal{B}(\mathrm{b}_n)$$

$$\mathcal{B}(\mathbf{c}) \leq \min_{w_1, \ldots, w_N} \mathcal{B}(w_1 \mathrm{b}_1 + \ldots + w_N \mathrm{b}_N)$$

Calibeating and Experts

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CALIBEATING is ***stronger*** !

Successful Economic Forecasting

TAKING PRIDE IN OUR RECORD

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"We have correctly forecasted 8 of the last 5 recessions"

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