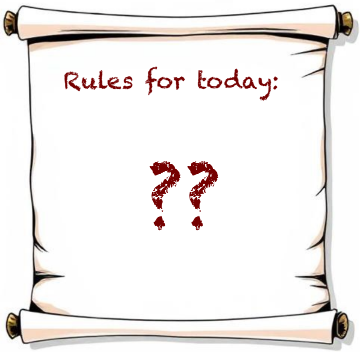
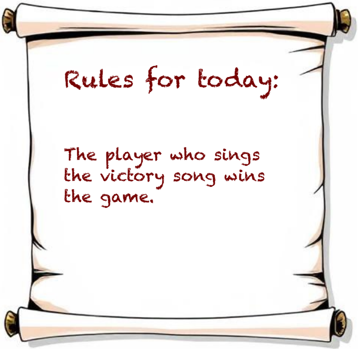
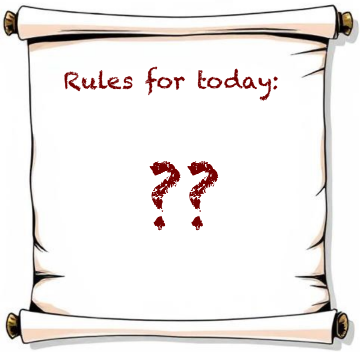
Calvinballs



Calvinballs



Calvinballs



Calvinballs

Calvinballs

Do Calvin and Hobbes ever have a chance of doing well in this game?

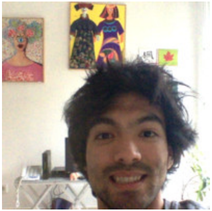
Calvinballs

Do Calvin and Hobbes ever have a chance of doing well in this game?

Yes! ... sometimes ... but they have to read our paper.

TRACKING SOLUTIONS OF TIME-VARYING VARIATIONAL INEQUALITIES

Based on joint work with



Hédi Hadiji

Cristóbal Guzmań

November 6, 2025

Overview

- Our objective: Time-varying Variational Inequalities.
- Special cases:
  - Dynamic Regret Guarantees for Online Learning;
  - Tracking Equilibria in Time-Varying Games;
- Tracking Guarantees for:
  - Tame Time-Varying Variational Inequalities;
  - Periodic Time-Varying Variational Inequalities.
- Negative Results.
- Open Questions.

Overview

- Our objective: Time-varying Variational Inequalities.
- Special cases:
  - Dynamic Regret Guarantees for Online Learning;
  - Tracking Equilibria in Time-Varying Games;
- Tracking Guarantees for:
  - Tame Time-Varying Variational Inequalities;
  - Periodic Time-Varying Variational Inequalities.
- Negative Results.
- Open Questions.

Variational Inequalities

Definition (Variational Inequality Problem)

Denote

- $\mathcal{C} \subseteq \mathbb{R}^d$, *closed and convex set,*
- $F : \mathcal{C} \to \mathcal{C}$, *L-Lipschitz continuous operator.*

The finite-dimensional Variational Inequality Problem $\mathrm{VIP}(F, \mathcal{C})$ *is defined as:*

$$\textit{find} \quad Z^\star \in \mathcal{C} \quad \textit{such that} \qquad \left\langle F(Z^\star), Z - Z^\star \right\rangle \geq 0 \quad \textit{for all} \quad Z \in \mathcal{C}\,. \tag{VIP}$$

Assumption: The VIP has a unique solution $Z^\star$.

Special Cases:

- Computing minimizers of convex optimization problems;
- Equilibrium computation in concave games.

Time-Varying Variational Inequality Problems

Protocol: For each iteration $t \in [T]$:

- Learner chooses $Z_t \in \mathcal{C}$ using algorithm $\mathcal{A}$;
- Nature chooses $F_t : \mathcal{C} \to \mathcal{C}$;
- Learner observes $F_t(Z_t)$.

Time-Varying Variational Inequality Problems

Protocol: For each iteration $t \in [T]$:

- Learner chooses $Z_t \in \mathcal{C}$ using algorithm $\mathcal{A}$;
- Nature chooses $F_t : \mathcal{C} \to \mathcal{C}$;
- Learner observes $F_t(Z_t)$.

Definition (Tracking Error)

***Tracking Error**: Let $Z_t^\star \in \mathcal{C}$ denote the solution for* $\mathrm{VIP}(F_t, \mathcal{C})$,

$$\tau_T(\mathcal{A}) = \sum_{t=1}^{T} \left\| Z_t - Z_t^\star \right\|^2 .$$

Lemma: For any algorithm $\mathcal{A}$, there exists a sequence (F_t), such that $\tau_T(\mathcal{A}) \gtrsim T$.

Special Case I: Dynamic Regret

Remark (Relation Tracking Error and Dynamic Regret.)

Dynamic regret:

$$\operatorname{dynReg}_T\left((Z_t^\star)_{t\in[T]}\right) = \sum_{t=1}^{T} f_t(Z_t) - f_t(Z_t^\star)\,.$$

Assume: *For all $t \in [T]$, f_t's are*

- *μ-strongly convex,*
- *differentiable,*
- *∇f_t are L-Lipschitz, and*
- $Z_t^\star \in \operatorname{int}(\mathcal{C})$.

Then

$$\frac{\mu}{2}\tau_T(\mathcal{A}) \leq \operatorname{dynReg}_T\left((Z_t^\star)_{t\in[T]}\right) \leq \frac{L}{2}\tau_T(\mathcal{A})\,.$$

Tame Time-Varying VIP

Definition (Tame Time-Varying VIP)

We call a time-varying VIP α-*tame* if the quadratic path length is sublinear: For $\alpha \in [0, 1)$

$$P_T^\star = \sum_{t=2}^{T} \left\| Z_{t-1}^\star - Z_t^\star \right\|^2 \lesssim T^\alpha .$$

Existing results:
Definition inspired by Duvocelle et al. [DMSV23].

Special Case II: Equilibrium Tracking for Time-varying Games

Two-Player Zero-Sum Time-Varying Games (Duvocelle et al. [DMSV23]):

Protocol: For each iteration $t \in [T]$:

- Agents chooses $x_t \in \mathcal{X}$ and $y_t \in \mathcal{Y}$ using algorithm $\mathcal{A}$;
- Nature chooses game $\nu_t : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$;
- Agents observes first-order feedback based on $\nu_t(x_t, y_t)$.

Lemma

- $\nu : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ *the differentiable and convex-concave.*
- $F([x, y]^\top) := [\nabla_x \nu(x, y), -\nabla_y \nu(x, y)]$

$$(x^\star, y^\star) \text{ is a Nash-equilibrium} \quad \Leftrightarrow \quad [x^\star, y^\star]^\top \text{ is a solution for } \mathrm{VIP}\,(F, \mathcal{X} \times \mathcal{Y}) \ .$$

Tracking Guarantees for Tame VIP

Definition (Contractive Algorithms)

Let $C \in (0,1)$. An algorithm $\mathcal{A}$ is said to be *C-contractive over a set of operators* $\mathcal{F}$ if for all $F \in \mathcal{F}$

$$\forall Z \in \mathcal{C}: \qquad \left\|\mathcal{A}(Z,F) - Z^\star\right\| \leq C \left\|Z - Z^\star\right\| ,$$

where $Z^\star$ is the solution for $\mathrm{VIP}(F, \mathcal{C})$.

Tracking Guarantees for Tame VIP

Definition (Contractive Algorithms)

Let $C \in (0,1)$. An algorithm $\mathcal{A}$ is said to be *C-contractive over a set of operators* $\mathcal{F}$ if for all $F \in \mathcal{F}$

$$\forall Z \in \mathcal{C} : \qquad \left\|\mathcal{A}(Z,F) - Z^\star\right\| \leq C \left\|Z - Z^\star\right\| ,$$

where $Z^\star$ is the solution for $\mathrm{VIP}(F, \mathcal{C})$.

Example:

- $\mathcal{A}$ is projected gradient descent with step-size c;
- $\mathcal{F} := \{\nabla f \mid f : \mathcal{C} \to \mathbb{R}$ and f is L-smooth and μ-strongly convex with $\frac{\mu}{L^2} \leq c\}$;
- $\mathcal{A}$ is $(1 - (\mu/L)^2)$ contractive over $\mathcal{F}$.

Tracking Guarantees for Tame VIP

Definition (Contractive Algorithms)

Let $C \in (0,1)$. An algorithm $\mathcal{A}$ is said to be *C-contractive over a set of operators* $\mathfrak{F}$ if for all $F \in \mathfrak{F}$

$$\forall Z \in \mathcal{C}: \qquad \left\|\mathcal{A}(Z,F) - Z^\star\right\| \leq C \left\|Z - Z^\star\right\| ,$$

where $Z^\star$ is the solution for $\mathrm{VIP}(F, \mathcal{C})$.

Theorem (Theorem 2.1 [HSG24])

Suppose $\mathcal{A}$ is C-contractive over $\mathfrak{F}$. Then, for any sequence of operators in $\mathfrak{F}$ with solutions $(Z_t^\star)_{t\in[T]}$, the tracking error is bounded by

$$\tau_T(\mathcal{A}) \lesssim \frac{1}{(1-C)^2} P_T^\star + 1 \, .$$

Tracking Guarantees for Tame VIP

Definition (Contractive Algorithms)

Let $C \in (0,1)$. An algorithm $\mathcal{A}$ is said to be *C-contractive over a set of operators* $\mathfrak{F}$ if for all $F \in \mathfrak{F}$

$$\forall Z \in \mathcal{C}: \qquad \left\|\mathcal{A}(Z,F) - Z^\star\right\| \le C \left\|Z - Z^\star\right\| ,$$

where $Z^\star$ is the solution for $\mathrm{VIP}(F, \mathcal{C})$.

Theorem (Theorem 2.1 [HSG24])

Suppose $\mathcal{A}$ is C-contractive over $\mathfrak{F}$. Then, for any sequence of operators in $\mathfrak{F}$ with solutions $(Z_t^\star)_{t\in[T]}$, the tracking error is bounded by

$$\tau_T(\mathcal{A}) \lesssim \frac{1}{(1-C)^2} P_T^\star + 1\,.$$

- This bound is tight up to constants.

Tracking Guarantees for Tame VIP

Definition (Contractive Algorithms)

Let $C \in (0,1)$. An algorithm $\mathcal{A}$ is said to be *C-contractive over a set of operators $\mathfrak{F}$* if for all $F \in \mathfrak{F}$

$$\forall Z \in \mathcal{C}: \qquad \left\|\mathcal{A}(Z,F) - Z^\star\right\| \leq C \left\|Z - Z^\star\right\| ,$$

where $Z^\star$ is the solution for $\mathrm{VIP}(F, \mathcal{C})$.

Theorem (Theorem 2.1 [HSG24])

Suppose $\mathcal{A}$ is C-contractive over $\mathfrak{F}$. Then, for any sequence of operators in $\mathfrak{F}$ with solutions $(Z_t^\star)_{t\in[T]}$, the tracking error is bounded by

$$\tau_T(\mathcal{A}) \lesssim \frac{1}{(1-C)^2} P_T^\star + 1 \,.$$

- This bound is tight up to constants.
- **Corollary:** For α-tame problems, this gives $\tau_T(\mathcal{A}) \lesssim T^\alpha$.

Comparison to Existing Results

	Rate	Setting
Duvocelle et al. [DMSV23]	$T^{\frac{2+\alpha}{3}}$	Stochastic feedback Strongly monotone time-varying games.
Besbes et al. [BGZ15]	$T^{\frac{1+\alpha}{2}}$	Stochastic feedback Strongly convex time-varying functions.
Mokhtari et al. [MSJR16]	T^{α}	Deterministic feedback Time-varying strongly convex functions.
Our Result	T^{α}	Deterministic feedback Time-varying VIPs - any contractive algorithm.

Open Question

- Can our results be generalized for stochastic feedback?
- Can the work by Duvocelle et al. [DMSV23] or Besbes et al. [BGZ15] be generalized for VIP?

Periodic Time-Varying Variational Inequality Problems

Definition (Periodic Time-Varying Variational Inequality Problems)

A time-varying variational inequality problem $\mathrm{VIP}((F_t), \mathcal{C})$ is periodic if there exists $k \in \mathbb{N}$, such that

$$F_{t+k} = F_t$$

for all $t \in \mathbb{N}$.

- **Example:** Seasonal shifting market.

Periodic Time-Varying Variational Inequality Problems

Definition (Periodic Time-Varying Variational Inequality Problems)

A time-varying variational inequality problem $\mathrm{VIP}((F_t), \mathcal{C})$ is periodic if there exists $k \in \mathbb{N}$, such that

$$F_{t+k} = F_t$$

for all $t \in \mathbb{N}$.

- **Example:** Seasonal shifting market.
- **Remark:** Periodic Time-Varying VIPs are not tame.

Periodic Time-Varying Variational Inequality Problems

Definition (Periodic Time-Varying Variational Inequality Problems)

A time-varying variational inequality problem $\mathrm{VIP}((F_t), \mathcal{C})$ is periodic if there exists $k \in \mathbb{N}$, such that

$$F_{t+k} = F_t$$

for all $t \in \mathbb{N}$.

- **Example:** Seasonal shifting market.
- **Remark:** Periodic Time-Varying VIPs are not tame.

Assumption (Strong Monotonicity)

Operator F is strongly monotone if

$$\exists \mu > 0 : \forall Z, Z' \in \mathcal{C} : \quad \left\langle F(Z) - F(Z'), Z - Z' \right\rangle \geq \mu \left\| Z - Z' \right\|^2 .$$

Cyclic Forward Method

Algorithm 1: $\mathcal{A}^{(k)}_{\mathrm{cFW}}$, Cyclic Forward Method

Input: Period length k, step-size schedule (η_t).

Initialization: Initial iterates $(Z_{1,1}, Z_{2,1}, \ldots, Z_{k,1}) = (Z_1, Z_1, \ldots, Z_1)$ with $Z_1 \in \mathcal{C}$.

for $t = 1, \ldots, T$ **do**

- Pick current index $n = (t \bmod k) + 1$;
- Play $Z_t = Z_{n,t}$, receive $F_t(Z_t)$;
- Update for all $i \in \{1, \ldots, k\}$

$$Z_{i,t+1} = \begin{cases} \mathrm{Proj}_{\mathcal{C}}(Z_t - \eta_t F_t(Z_t)) & \text{for } i = n \\ Z_{i,t} & \text{otherwise.} \end{cases}$$

Tracking Guarantees

Theorem (Corollary 3.1 [HSG24])

Let (F_t) be a k-periodic sequence. Assume

- *F_t's are μ-strongly monotone operators,*
- *there exists a constant G such that $\forall Z \in \mathcal{C} : \|F_t(Z)\| \leq G$.*

Set $\eta_t = k/(\mu t)$. Then

$$\tau_T(\mathcal{A}_{\mathrm{cFW}}^{(k)}) \lesssim k \left(\frac{G}{\mu}\right)^2 \left(\log\left(\frac{T}{k}\right) + 1\right) .$$

Problem: We do not necessarily know k.

Meta-Algorithm: Intuition

Suppose we know an upper bound $K \geq k$.

$$\mathcal{A}^{(1)}_{\mathrm{cFW}}, \quad \mathcal{A}^{(2)}_{\mathrm{cFW}}, \quad \cdots \quad \mathcal{A}^{(k-1)}_{\mathrm{cFW}}, \quad \mathcal{A}^{(k)}_{\mathrm{cFW}}, \quad \mathcal{A}^{(k+1)}_{\mathrm{cFW}}, \quad \cdots \quad \mathcal{A}^{(K-1)}_{\mathrm{cFW}}, \quad \mathcal{A}^{(K)}_{\mathrm{cFW}}$$

Meta-Algorithm: Intuition

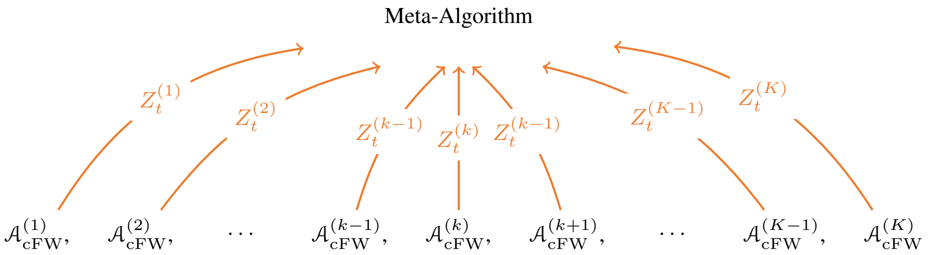
Suppose we know an upper bound $K \geq k$.

Meta-Algorithm

$\mathcal{A}^{(1)}_{\text{cFW}}, \quad \mathcal{A}^{(2)}_{\text{cFW}}, \quad \cdots \quad \mathcal{A}^{(k-1)}_{\text{cFW}}, \quad \mathcal{A}^{(k)}_{\text{cFW}}, \quad \mathcal{A}^{(k+1)}_{\text{cFW}}, \quad \cdots \quad \mathcal{A}^{(K-1)}_{\text{cFW}}, \quad \mathcal{A}^{(K)}_{\text{cFW}}$

Meta-Algorithm: Intuition

Suppose we know an upper bound $K \geq k$.



Meta-Algorithm: Intuition

Suppose we know an upper bound $K \geq k$.

Nature

Meta-Algorithm

Compute Z_t, based on $Z_t^{(1)}, \ldots, Z_t^{(K)}$

$\mathcal{A}_{\mathrm{cFW}}^{(1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(2)}, \quad \cdots \quad \mathcal{A}_{\mathrm{cFW}}^{(k-1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(k)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(k+1)}, \quad \cdots \quad \mathcal{A}_{\mathrm{cFW}}^{(K-1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(K)}$

Meta-Algorithm: Intuition

Suppose we know an upper bound $K \geq k$.

$$\begin{tikzcd}
\text{Nature} \arrow[d, bend left, "F_t(Z_t)"] \\
\text{Meta-Algorithm} \arrow[u, bend left, "Z_t"]
\end{tikzcd}$$

$$\mathcal{A}_{\mathrm{cFW}}^{(1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(2)}, \quad \cdots \quad \mathcal{A}_{\mathrm{cFW}}^{(k-1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(k)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(k+1)}, \quad \cdots \quad \mathcal{A}_{\mathrm{cFW}}^{(K-1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(K)}$$

Meta-Algorithm: Intuition

Suppose we know an upper bound $K \geq k$.

Nature

Meta-Algorithm

$\tilde{F}_t$

$$\mathcal{A}_{\mathrm{cFW}}^{(1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(2)}, \quad \ldots \quad \mathcal{A}_{\mathrm{cFW}}^{(k-1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(k)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(k+1)}, \quad \ldots \quad \mathcal{A}_{\mathrm{cFW}}^{(K-1)}, \quad \mathcal{A}_{\mathrm{cFW}}^{(K)}$$

Meta-Algorithm

Algorithm 2: Meta-Algorithm $\mathcal{A}_{\mathrm{meta}}$

Data: Maximum period length $K \in \mathbb{N}$,
prior distribution $p_1 \in \Delta_K$ over expert algorithms.

for t = *1*, ..., T **do**

Receive $(Z_t^{(1)}, \ldots, Z_t^{(K)})$ from expert algorithms $\mathcal{A}_{cFW}^{(1)}, \ldots, \mathcal{A}_{cFW}^{(K)}$;

Play $Z_t = p_{t,1} Z_t^{(1)} + \cdots + p_{t,K} Z_t^{(K)}$;

Receive $F_t(Z_t)$;

Update p_t using exponential weights;

Send surrogate operator $\tilde{F}_t$ to experts $\mathcal{A}_{cFW}^{(1)}, \ldots, \mathcal{A}_{cFW}^{(K)}$;

Remark: The algorithm only requires one evaluation of F_t.

Tracking Guarantees

Theorem (Corollary 3.2 [HSG24])

Let (F_t) be a k-periodic sequence. Assume

- *F_t's are μ-strongly monotone operators,*
- $\operatorname{diameter}(\mathcal{C})$ *is bounded, and*
- $\forall Z \in \mathcal{C} : \|F_t(Z)\| \leq G$.

If upper bound $K \geq k$ is known, then

$$\tau_T(\mathcal{A}_{\text{meta}}) \lesssim \kappa^2 D^2 \left(k \left(\log \left(\frac{T}{k} \right) + 1 \right) + \log K \right),$$

where $\kappa = L/\mu$.

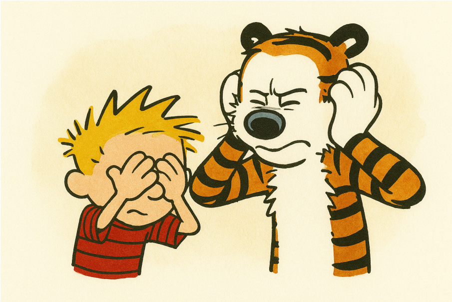
Remark: Tracking bound does not depend on path length $P_T^\star$.

Open Research Questions

- Can we have similar results for periodic time-varying VIPs with stochastic feedback?
- Can we obtain tracking guarantees for periodic time-varying VIPs with a truly distributed meta-algorithm?
- Can we show tracking guarantees for 'almost' periodic problems?

Let's Ignore the Theory

Can we just ignore time variance?

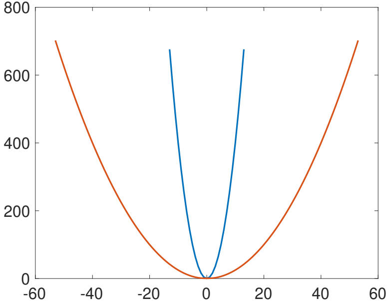


Negative Results: Setting

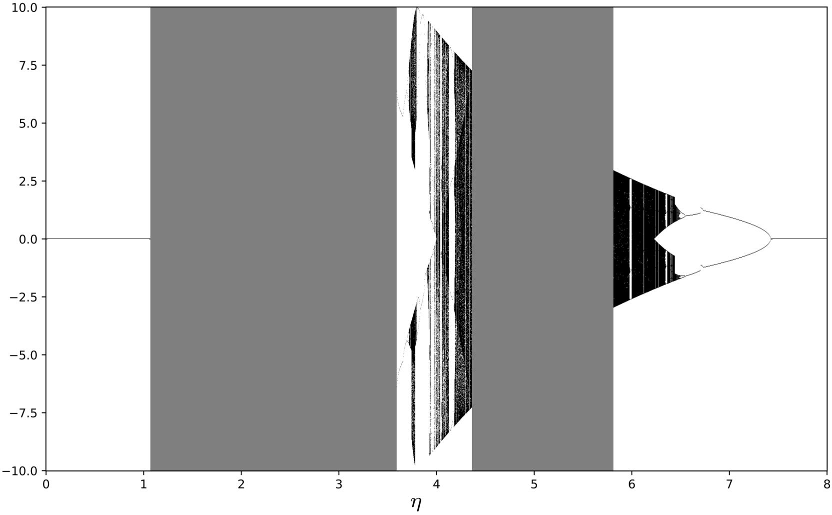
$$\forall t \in 2\,\mathbb{N} : f_t(x) = \log\left(1 + e^{4x^2/2}\right),$$
$$\forall t \notin 2\,\mathbb{N} : f_t(x) = \log\left(1 + e^{\frac{1}{4}x^2/2}\right).$$

We use gradient descent:

$$x \mapsto x - \eta f_t'(x).$$



Negative Results



References

Omar Besbes, Yonatan Gur, and Assaf Zeevi.
Non-stationary stochastic optimization.
Operations Research, 63(5):1227–1244, 2015.

Benoit Duvocelle, Panayotis Mertikopoulos, Mathias Staudigl, and Dries Vermeulen.
Multiagent online learning in time-varying games.
Mathematics of Operations Research, 48(2):914–941, may 2023.

Hédi Hadiji, Sarah Sachs, and Cristóbal Guzmán.
Tracking Solutions of Time-Varying Variational Inequalities.
(to appear in *Mathematics of Operations Research*), 2024.

Aryan Mokhtari, Shahin Shahrampour, Ali Jadbabaie, and Alejandro Ribeiro.
Online optimization in dynamic environments: Improved regret rates for strongly convex problems.
In *2016 IEEE 55th Conference on Decision and Control (CDC)*, pages 7195–7201. IEEE, 2016.

Appendix: Proof of Theorem 2.1 [HSG24]

Use $\|u+v\|^2 \leq (1+\alpha)\|u\|^2 + (1+\alpha^{-1})\|v\|^2$ for any $u, v \in \mathbb{R}^d$ and $\alpha > 0$:

$$\underbrace{\left\|Z_{t+1} - Z^\star_{t+1}\right\|^2}_{=\|u+v\|^2} \leq \left(1 + \underbrace{\left(\frac{1}{C} - 1\right)}_{=\alpha}\right) \underbrace{\left\|Z_{t+1} - Z^\star_t\right\|^2}_{=\|u\|^2} + \left(1 + \frac{1}{1/C - 1}\right) \underbrace{\left\|Z^\star_{t+1} - Z^\star_t\right\|^2}_{=\|v\|^2}$$

Appendix: Proof of Theorem 2.1 [HSG24]

Use $\|u+v\|^2 \leq (1+\alpha)\|u\|^2 + (1+\alpha^{-1})\|v\|^2$ for any $u, v \in \mathbb{R}^d$ and $\alpha > 0$:

$$\underbrace{\left\|Z_{t+1} - Z^\star_{t+1}\right\|^2}_{=\|u+v\|^2} \leq \left(1 + \underbrace{\left(\frac{1}{C} - 1\right)}_{=\alpha}\right) \underbrace{\left\|Z_{t+1} - Z^\star_t\right\|^2}_{=\|u\|^2} + \left(1 + \frac{1}{1/C - 1}\right) \underbrace{\left\|Z^\star_{t+1} - Z^\star_t\right\|^2}_{=\|v\|^2}$$

$$= \frac{1}{C}\left\|Z_{t+1} - Z^\star_t\right\|^2 + \frac{1}{1-C}\left\|Z^\star_{t+1} - Z^\star_t\right\|^2$$

Appendix: Proof of Theorem 2.1 [HSG24]

Use $\|u+v\|^2 \le (1+\alpha)\|u\|^2 + (1+\alpha^{-1})\|v\|^2$ for any $u, v \in \mathbb{R}^d$ and $\alpha > 0$:

$$\begin{aligned}
\underbrace{\left\|Z_{t+1} - Z_{t+1}^\star\right\|^2}_{=\|u+v\|^2} &\le \left(1 + \underbrace{\left(\frac{1}{C} - 1\right)}_{=\alpha}\right) \underbrace{\left\|Z_{t+1} - Z_t^\star\right\|^2}_{=\|u\|^2} + \left(1 + \frac{1}{1/C - 1}\right) \underbrace{\left\|Z_{t+1}^\star - Z_t^\star\right\|^2}_{=\|v\|^2} \\
&= \frac{1}{C}\left\|Z_{t+1} - Z_t^\star\right\|^2 + \frac{1}{1-C}\left\|Z_{t+1}^\star - Z_t^\star\right\|^2 \\
&\le C\left\|Z_t - Z_t^\star\right\|^2 + \frac{1}{1-C}\left\|Z_{t+1}^\star - Z_t^\star\right\|^2 .
\end{aligned}$$

Appendix: Proof of Theorem 2.1 [HSG24]

Use $\|u+v\|^2 \leq (1+\alpha)\|u\|^2 + (1+\alpha^{-1})\|v\|^2$ for any $u, v \in \mathbb{R}^d$ and $\alpha > 0$:

$$\begin{aligned}
\underbrace{\left\|Z_{t+1} - Z_{t+1}^\star\right\|^2}_{=\|u+v\|^2} &\leq \left(1 + \underbrace{\left(\frac{1}{C} - 1\right)}_{=\alpha}\right) \underbrace{\left\|Z_{t+1} - Z_t^\star\right\|^2}_{=\|u\|^2} + \left(1 + \frac{1}{1/C - 1}\right) \underbrace{\left\|Z_{t+1}^\star - Z_t^\star\right\|^2}_{=\|v\|^2} \\
&= \frac{1}{C}\left\|Z_{t+1} - Z_t^\star\right\|^2 + \frac{1}{1-C}\left\|Z_{t+1}^\star - Z_t^\star\right\|^2 \\
&\leq C\left\|Z_t - Z_t^\star\right\|^2 + \frac{1}{1-C}\left\|Z_{t+1}^\star - Z_t^\star\right\|^2 .
\end{aligned}$$

Summing over $T-1$ steps:

$$\sum_{t=1}^{T-1} \|Z_{t+1} - Z_{t+1}^\star\|^2 \leq C \sum_{t=1}^{T-1} \|Z_t - Z_t^\star\|^2 + \frac{1}{1-C} \sum_{t=1}^{T-1} \|Z_{t+1}^\star - Z_t^\star\|^2 .$$

Appendix: Proof of Theorem 2.1 [HSG24]

Use $\|u+v\|^2 \leq (1+\alpha)\|u\|^2 + (1+\alpha^{-1})\|v\|^2$ for any $u, v \in \mathbb{R}^d$ and $\alpha > 0$:

$$\begin{aligned}
\underbrace{\left\|Z_{t+1} - Z_{t+1}^\star\right\|^2}_{=\|u+v\|^2} &\leq \left(1 + \underbrace{\left(\frac{1}{C} - 1\right)}_{=\alpha}\right) \underbrace{\left\|Z_{t+1} - Z_t^\star\right\|^2}_{=\|u\|^2} + \left(1 + \frac{1}{1/C - 1}\right) \underbrace{\left\|Z_{t+1}^\star - Z_t^\star\right\|^2}_{=\|v\|^2} \\
&= \frac{1}{C}\left\|Z_{t+1} - Z_t^\star\right\|^2 + \frac{1}{1-C}\left\|Z_{t+1}^\star - Z_t^\star\right\|^2 \\
&\leq C\left\|Z_t - Z_t^\star\right\|^2 + \frac{1}{1-C}\left\|Z_{t+1}^\star - Z_t^\star\right\|^2 .
\end{aligned}$$

Summing over $T-1$ steps:

$$\sum_{t=1}^{T-1} \|Z_{t+1} - Z_{t+1}^\star\|^2 \leq C \sum_{t=1}^{T-1} \|Z_t - Z_t^\star\|^2 + \frac{1}{1-C} \sum_{t=1}^{T-1} \|Z_{t+1}^\star - Z_t^\star\|^2 .$$

Reorganizing and adding $\|Z_1 - Z_1^\star\|^2$:

$$\sum_{t=1}^{T} \|Z_t - Z_t^\star\|^2 \leq \frac{1}{(1-C)^2} \sum_{t=1}^{T-1} \|Z_{t+1}^\star - Z_t^\star\|^2 + \frac{1}{1-C}\|Z_1 - Z_1^\star\|^2 .$$

Appendix: Details on Surrogate Loss

Aggregator Method:

- Use exponential weights updates:

$$p_{t,i} \propto \exp\left(-\lambda \sum_{s=1}^{t-1} \ell_{s,i}\right).$$

- Obtain loss

$$\ell_{s,i} = \langle F_s(Z_s), Z_s^{(i)} \rangle + \frac{\mu}{2} \left\| Z_s^{(i)} - Z_s \right\|^2 .$$

Surrogate Operator $\tilde{F}_t$:

$$\tilde{F}_t : z \mapsto F_t(Z_t) + \mu(z - Z_t) .$$

Appendix: Distributed Setting

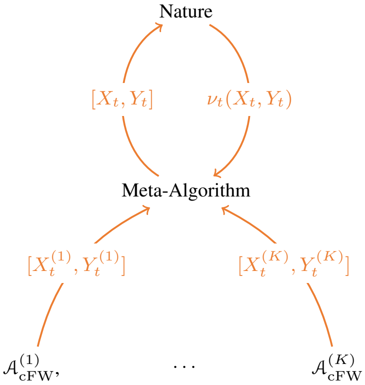
Open Question:

Can we obtain tracking guarantees for periodic VIPs with a distributed meta-algorithm?

Appendix: Distributed Setting

Open Question:

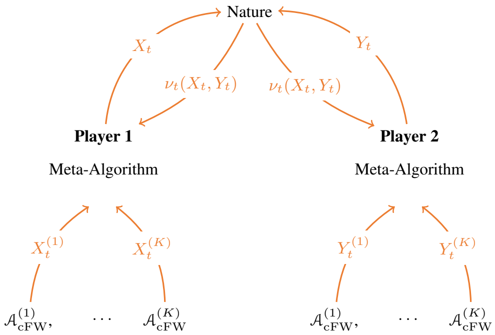
Can we obtain tracking guarantees for periodic VIPs with a distributed meta-algorithm?



Appendix: Distributed Setting

Open Question:

Can we obtain tracking guarantees for periodic VIPs with a distributed meta-algorithm?



Appendix: Let's Ignore the Theory

Can we just ignore time variance?