ON THE SUPREMUM OF CONVEX LOWER SEMICONTINUOUS FUNCTIONS ON COMPACT DOMAINS

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Abstract. In finite dimension, we consider convex lower semicontinuous functions with nonempty compact domain and address the following questions. (i) Is such a function necessarily bounded? (ii) If bounded on its domain, does such a function necessarily attain a maximum on its domain? We answer both questions in the negative by providing counterexamples that share a similar construction.

1. Introduction

The extreme value theorem ensures that a continuous function admits both a minimum and a maximum on a nonempty compact set, and is attributed to Bolzano—see e.g. [Rud76, Theorem 4.16] for a textbook reference. This can be adapted to lower semicontinuous functions: they necessarily admit a minimum on a nonempty compact set—see e.g. [Bou07, §6, Théorème 3]. In this note, we focus on *convex* lower semicontinuous functions on nonempty compact domains.

Definition 1. Let $n \geqslant 1$ and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. The *domain* of f is the set of points where the function has finite values:

$$\operatorname{dom} f = \{x \in \mathbb{R}^n,\ f(x) < +\infty\}.$$

f is *lower semicontinuous* if for all $\alpha \in \mathbb{R}$, the set $\{x \in \mathbb{R}^n,\ f(x) \leqslant \alpha\}$ is closed.

One characterization of lower semicontinuity is the following: when x converges to a point x_0, the value $f(x)$ cannot jump *up* at the limit:

$$\liminf_{x \to x_0} f(x) \geqslant f(x_0).$$

Besides, one intuitive consequence of convexity seems to be the following: the function cannot jump *down* when converging to the boundary of the domain from the inside. Combining the two leads one to believe that the function is necessarily continuous on its domain. If the domain is compact, this would in turn guarantee that a maximum is attained. However, the functions constructed below show that this intuition is wrong: it is possible for the function to jump down when reaching the boundary of the domain from the inside, while preserving convexity.

We consider the following questions regarding convex lower semicontinuous functions with nonempty compact domains. Is such a function necessarily bounded on its domain? And in the bounded case, does such a function necessarily admit a maximum on its domain? We construct counterexamples for both questions.

Note that these questions become trivial if either convexity or lower semicontinuity is relaxed. Indeed, dropping lower semicontinuity, a function with value 0 in the

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interior of a Euclidean ball and arbitrary finite values on the boundary is convex, and therefore can either be unbounded or bounded with no maximum attained on its domain. If we instead drop convexity, functions $f, g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x} & \text{if } 0 < x \leqslant 1, \\ +\infty & \text{otherwise,} \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 - x & \text{if } 0 < x \leqslant 1, \\ +\infty & \text{otherwise,} \end{cases}$$

are lower semicontinuous, have domain $[0, 1]$, f is not bounded on its domain, and g is bounded on its domain but does not attain a maximum on it.

A somewhat related result is the Gale–Klee–Rockafellar theorem [GKR68], which ensures that a bounded convex function on the relative interior of a convex *polytope* can be extended in a unique way to a continuous convex function on the whole polytope, ensuring the existence of a maximum.

2. Counterexamples

Let $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x} & \text{if } x > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

In particular, f is nonnegative.

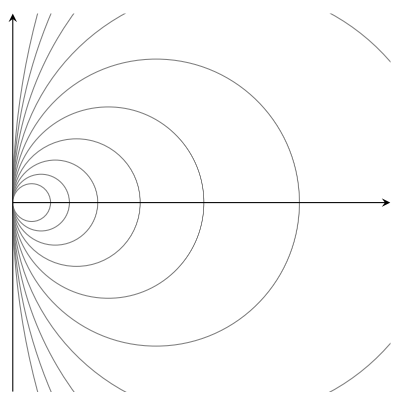

Figure 1. Level lines of f.

Lemma 2. *f is convex.*

Proof. f is twice differentiable on $\mathbb{R}_+^* \times \mathbb{R}$ and its Hessian writes, for $x > 0$ and $y \in \mathbb{R}$,

$$\nabla^2 f(x, y) = \begin{pmatrix} 2y^2/x^3 & -2y/x^2 \\ -2y/x^2 & 2/x \end{pmatrix}.$$

The first coefficient and the determinant are nonnegative, which is enough to deduce that this 2×2 symmetric matrix is positive semi-definite. This is true for all $(x, y) \in \mathbb{R}_+^* \times \mathbb{R}$. f is thus convex on $\mathbb{R}_+^* \times \mathbb{R}$ which is its domain. $\square$

Let $g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$g(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ f(x,y) & \text{otherwise.} \end{cases}$$

g only differs from f at $(0,0)$ where $g(0,0) = 0$, whereas $f(0,0) = +\infty$. In particular, g is also nonnegative.

Lemma 3. *g is convex and lower semicontinuous.*

Proof. Let $z, z' \in \mathbb{R}^2$ and $\lambda \in (0,1)$. We aim at proving the following convexity inequality:

$$g(\lambda z + (1-\lambda)z') \leqslant \lambda\, g(z) + (1-\lambda) g(z').$$

Denote $z = (x,y)$ and $z' = (x', y')$.

If either $x < 0$ or $x' < 0$, the right-hand side is infinite, and the inequality is true. If $x > 0$ and $x' > 0$, then the inequality is true because g and f coincide on $(\mathbb{R}_+^*)^2 \times \mathbb{R}$. We now assume $x = 0$ and $x' > 0$. If $y \neq 0$, $g(z) = g(x,y) = +\infty$, the right-hand side is infinite, and the inequality is true.

If $y = 0$, then $z = (0,0)$ and let $\varepsilon > 0$. Because $\varepsilon z' \in (\mathbb{R}_+^*)^2 \times \mathbb{R}$, we can use the already established cases to write

$$\begin{aligned} g(\lambda(\varepsilon z') + (1-\lambda)z') &\leqslant \lambda\, g(\varepsilon z') + (1-\lambda) g(z') \\ &= \lambda \varepsilon \frac{(x')^2 + (y')^2}{x'} + (1-\lambda) g(z'). \end{aligned}$$

As $\varepsilon \to 0^+$, $\lambda(\varepsilon z') + (1-\lambda)z'$ belongs to $(\mathbb{R}_+^*)^2 \times \mathbb{R}$ and converges to $(1-\lambda)z' \in (\mathbb{R}_+^*)^2 \times \mathbb{R}$. Therefore, the above left-hand side converges to $g((1-\lambda)z')$ because g coincide on $(\mathbb{R}_+^*)^2 \times \mathbb{R}$ with f, which is continuous on $(\mathbb{R}_+^*)^2 \times \mathbb{R}$. To the limit, the inequality becomes

$$g((1-\lambda)z') \leqslant (1-\lambda) g(z').$$

Using the fact that $z = (0,0)$ and $g(z) = 0$ by definition, the above indeed rewrites as

$$g(\lambda z + (1-\lambda)z') \leqslant \lambda\, g(z) + (1-\lambda) g(z').$$

The case $x > 0$ and $x' = 0$ is similar. Therefore, we have established the convexity inequality in all cases.

We now turn to lower semicontinuity. Let $\alpha \in \mathbb{R}$. If $\alpha < 0$,

$$\left\{(x,y) \in \mathbb{R}^2,\ g(x,y) \leqslant \alpha\right\} = \varnothing,$$

which is closed. If $\alpha \geqslant 0$,

$$\begin{aligned} \left\{(x,y) \in \mathbb{R}^2,\ g(x,y) \leqslant \alpha\right\} &= \{(0,0)\} \cup \left\{(x,y) \in \mathbb{R}^2,\ f(x,y) \leqslant \alpha\right\} \\ &= \{(0,0)\} \cup \left\{(x,y) \in \mathbb{R}_+^* \times \mathbb{R},\ \frac{x^2+y^2}{x} \leqslant \alpha\right\} \\ &= \{(0,0)\} \cup \left\{(x,y) \in \mathbb{R}_+^* \times \mathbb{R},\ \left(x - \frac{\alpha}{2}\right)^2 + y^2 \leqslant \frac{\alpha^2}{4}\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2,\ \left(x - \frac{\alpha}{2}\right)^2 + y^2 \leqslant \frac{\alpha^2}{4}\right\}, \end{aligned}$$

where we recognize the equation of a closed Euclidean ball. The set is thus closed, and g is lower semicontinous. $\square$

For $0 < a \leqslant 1$, let $\mathcal{C}_a \subset \mathbb{R}^2$ be defined as

$$\mathcal{C}_a = \left\{ (x,y) \in \left[0, \frac{\sqrt{5}-1}{2}\right] \times \mathbb{R},\ 0 \leqslant y \leqslant (x - x^2 - x^3)^a \right\}.$$

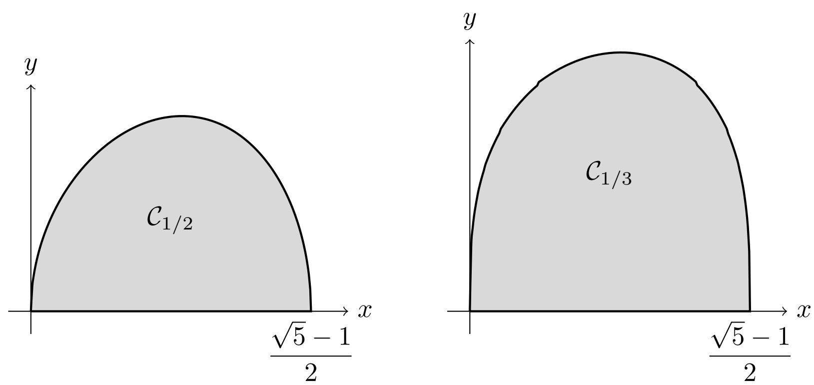

FIGURE 2. $\mathcal{C}_{1/2}$ and $\mathcal{C}_{1/3}$.

Lemma 4. *For all $0 < a \leqslant 1$, $\mathcal{C}_a$ is nonempty, convex and compact.*

Proof. Let us first prove that $\phi : x \mapsto (x - x^2 - x^3)^a$ is concave on $(0, \frac{\sqrt{5}-1}{2})$. It will then be concave on closed interval $[0, \frac{\sqrt{5}-1}{2}]$ by continuity. Let us also consider $\psi(x) = x - x^2 - x^3$. Functions ϕ and ψ are positive and twice differentiable on $(0, \frac{\sqrt{5}-1}{2})$, and for $x \in (0, \frac{\sqrt{5}-1}{2})$,

$$\psi''(x) = -2 - 6x < 0.$$

Then, simple computation yields that for $x \in (0, \frac{\sqrt{5}-1}{2})$,

$$\frac{\phi''(x)}{\phi(x)} = a\frac{\psi''(x)}{\psi(x)} + a(a-1)\frac{\psi'(x)^2}{\psi(x)^2}.$$

The above first term is negative because ψ'' is negative and ψ is positive. The second term is nonpositive because $0 < a \leqslant 1$. Therefore ψ'' is negative on $(0, \frac{\sqrt{5}-1}{2})$ and ψ is thus concave on $[0, \frac{\sqrt{5}-1}{2}]$.

$\mathcal{C}_a$ is nonempty as it contains $(0,0)$. It is obviously bounded. It can be written as the intersection of closed half-spaces with the hypograph of concave continuous function ϕ on interval $[0, \frac{\sqrt{5}-1}{2}]$, which is therefore closed and convex. $\mathcal{C}$ is therefore closed and convex. Hence the result. □

For $0 < a \leqslant 1$, let $h_a : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$h_a(x,y) = \begin{cases} g(x,y) & \text{if } (x,y) \in \mathcal{C}_a \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 5. *For $0 < a \leqslant 1$, h_a is convex, lower semicontinuous and* $\operatorname{dom} h_a = \mathcal{C}_a$.

Proof. h_a can be written as the sum of two convex lower semicontinuous functions: g and $I_{\mathcal{C}_a}$ the convex indicator of closed convex set $\mathcal{C}_a$. It is therefore convex and lower semicontinuous. We easily check that $\mathcal{C}_a \subset \operatorname{dom} g$ and we deduce that $\operatorname{dom} h_a = \mathcal{C}_a$. □

Theorem 6. *$h_{1/3}$ is a convex, lower semicontinuous function with nonempty compact domain and which is not bounded from above on its domain.*

Proof. For $k \geqslant 2$ integer, let

$$x_k = \frac{1}{k} \quad \text{and} \quad y_k = (x_k - x_k^2 - x_k^3)^{1/3},$$

so that (x_k, y_k) belongs to $\mathcal{C}_{1/3}$. Then, as $k \to +\infty$,

$$h_{1/3}(x_k, y_k) = \frac{\dfrac{1}{k^2} + \left(\dfrac{1}{k} - \dfrac{1}{k^2} - \dfrac{1}{k^3}\right)^{2/3}}{\dfrac{1}{k}} \sim k^{1/3}.$$

Hence, $h_{1/3}$ is not bounded from above on its domain. □

Theorem 7. *$h_{1/2}$ is a convex lower semicontinuous function with nonempty compact domain, which is bounded on its domain, and which does not admit a maximum on its domain.*

Proof. Let us prove that $h_{1/2}(\mathcal{C}_{1/2}) = [0, 1)$, this will imply that $h_{1/2}$ is bounded on its domain but does not admit a maximum on it.

Let $(x, y) \in \mathcal{C}_{1/2}$. If $x = 0$, necessarily $y = 0$ by definiton of $\mathcal{C}_{1/2}$ and

$$h_{1/2}(0, 0) = 0 \in [0, 1).$$

If $0 < x \leqslant \frac{\sqrt{5}-1}{2}$,

$$h_{1/2}(x, y) = \frac{x^2 + y^2}{x} \leqslant \frac{x^2 + x - x^2 - x^3}{x} = 1 - x^2 \in [0, 1).$$

This proves $h_{1/2}(\mathcal{C}_{1/2}) \subset [0, 1)$.

Conversely, let $t \in [0, 1)$. If $t = 0$, consider $(x, y) = (0, 0) \in \mathcal{C}$ which gives

$$h_{1/2}(0, 0) = 0.$$

If $t \geqslant \frac{\sqrt{5}-1}{2}$, let $x = \sqrt{1-t}$. Then, $0 < x \leqslant \frac{\sqrt{5}-1}{2}$ and let $y = \sqrt{x - x^2 - x^3}$ which ensures that $(x, y) \in \mathcal{C}_{1/2}$ and

$$h_{1/2}(x, y) = \frac{x^2 + y^2}{x} = 1 - x^2 = t.$$

In the case $0 < t \leqslant \frac{\sqrt{5}-1}{2}$, let $x = t$ and $y = 0$, which implies that $(x, y) \in \mathcal{C}_{1/2}$ and

$$h_{1/2}(x, y) = x = t.$$

This proves $h_{1/2}(\mathcal{C}_{1/2}) \supset [0, 1)$. □

We see in these two examples $h_{1/3}$ and $h_{1/2}$ that when following the upper boundary of the domain towards $(0, 0)$, the value of the function increases but jumps down to 0 when reaching point $(0, 0)$. When considering function g, which is convex and lower semicontinuous, following the same paths corresponds to the *interior of the domain*, and the value of g also jumps down to 0 when reaching the point $(0, 0)$.

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