Exercices
UMD Theory
Université Paris–Saclay

Let $d \geqslant 1$ and $\mathscr{X}$ a nonempty closed convex set in $\mathbb{R}^d$.

Exercice 1. — Let $\mathrm{F} : \mathbb{R}^d \to \mathbb{R}$ be a convex function. Prove that $\mathrm{F} + \mathrm{I}_{\mathscr{X}}$ is lower semicontinuous.

Exercice 2 (*Euclidean regularizer*). — Let h_2 be defined as

$$h_2(x) = \frac{1}{2} \|x\|_2^2 + \mathrm{I}_{\mathscr{X}}(x), \qquad x \in \mathbb{R}^d.$$

1) Prove that h_2 is a regularizer, satisfies $\operatorname{dom} h_2 = \mathscr{X}$ and that

$$\nabla h_2^*(y) = \operatorname*{arg\,min}_{x \in \mathscr{X}} \|y - x\|, \qquad y \in \mathbb{R}^d.$$

2) Let $(u_t)_{t \geqslant 1}$ a sequence in $\mathbb{R}^d$ and $x_1 \in \mathscr{X}$.

   a) Define $x_{t+1} = \Pi_{\mathscr{X}}(x_t + u_t)$ for all $t \geqslant 1$. Prove that $((x_t, x_t))_{t \geqslant 1}$ is a sequence of strict UMD iterates associated with h_2 and $(u_t)_{t \geqslant 1}$.

   b) Define $y_1 = x_1$, $y_{t+1} = y_t + u_t$ and $x_{t+1} = \Pi_{\mathscr{X}}(y_{t+1})$ for all $t \geqslant 1$. Prove that $((x_t, y_t))_{t \geqslant 1}$ is a sequence of strict UMD iterates associated with h_2 and $(u_t)_{t \geqslant 1}$.

Exercice 3 (*Entropic regularizer*). — Let $h_{\text{ent}} : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$h_{\text{ent}}(x) = \begin{cases} \sum_{i=1}^d x_i \log x_i & \text{if } x \in \Delta_d \\ +\infty & \text{otherwise,} \end{cases}$$

with convention $0 \log 0 = 0$.

1) Prove that h_{ent} is a regularizer and that $\operatorname{dom} h_{\text{ent}} = \Delta_d$.

2) Compute $h_{\text{ent}} - \min h_{\text{ent}}$.

3) Prove that

$$\nabla h_{\text{ent}}^*(y) = \left(\frac{\exp(y_i)}{\sum_{j=1}^d \exp(y_j)} \right)_{1 \leqslant i \leqslant d}, \qquad y \in \mathbb{R}^d.$$

4) Express the Bregman divergence associated with h_{ent}.

5) Prove that h_{ent} is 1-strongly convex with respect to $\|\cdot\|_1$.

Exercice 4 (*ℓ^p regularizer*). — Let $p \in (1,2)$ and $h_p : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be defined as

$$h_p(x) = \frac{1}{2} \|x\|_p^2 + \mathrm{I}_{\mathscr{X}}(x), \qquad y \in \mathbb{R}^d.$$

1) Prove that h_p is a regularizer.

2) In the case $\mathscr{X} = \mathbb{R}^d$, prove that h_p is twice differentiable, compute its gradient and hessian matrix, and express h_p^*.

3) In the case $\mathscr{X} = (\mathbb{R}_+)^d$, express h_p^* and ∇h_p^*.

4) In the general case, prove that h_p is $(p-1)$-strongly convex with respect to $\|\cdot\|_p$.