Exercices
Online convex optimization
Université Paris–Saclay

Let $d \geqslant 1$ be an integer and $\mathscr{X}$ a nonempty closed convex subset of $\mathbb{R}^d$.

Exercice 1 (*Online linear classification with absolute loss*). — Let $\mathscr{W} \subset \mathbb{R}^d$ be a nonempty set such that

$$\forall x \in \mathscr{X},\ \forall w \in \mathscr{W}, \qquad |\langle w, x \rangle| \leqslant 1.$$

We consider the following online linear classification problem. At step $t \geqslant 1$,

- Nature chooses and reveals $w_t \in \mathscr{W}$,
- the Decision Maker chooses $x_t \in \mathscr{X}$
- Nature chooses $z_t \in \{-1, 1\}$
- draw $\hat{z}_t = \begin{cases} 1 & \text{with probability } \frac{\langle w_t, x_t \rangle + 1}{2} \\ -1 & \text{with probability } \frac{1 - \langle w_t, x_t \rangle}{2}, \end{cases}$
- Nature reveals z_t and the Decision Maker incurs loss $|\hat{z}_t - z_t|$.

1) By considering expectations, explain how the above problem can be reduced to a deterministic problem with convex loss functions $\ell_t(x) = |\langle w_t, x \rangle - z_t|$.

2) Propose at least two algorithms for this problem and derive corresponding regret guarantees.

Exercice 2 (*Alternatives to FTRL and FTL*). — Let $(\ell_t)_{t\geqslant 1}$ be differentiable losses on $\mathcal{X}$, H be a mirror map compatible with $\mathcal{X}$, $x_0 \in \mathcal{X} \cap \operatorname{int}\operatorname{dom} H$ and consider

$$x_{t+1} = \operatorname*{arg\,min}_{x\in\mathcal{X}} \left\{ \langle \nabla \ell_t(x_t), x\rangle + \sum_{s=1}^{t} D_{\ell_s}(x, x_t) + D_H(x, x_t) \right\}.$$

1) Prove that the above can be interpreted as UMD iterates.

2) Derive a regret bound (on $\sum_{t=1}^{T}(\ell_t(x_t) - \ell_t(x))$) and compare with the corresponding regret bound for FTRL.

3) Generalize with a sequence of time-dependent mirror maps $(H_t)_{t\geqslant 0}$ and derive similar results as for FTRL.

4) In the case of differentiable strongly convex losses, consider

$$x_{t+1} = \operatorname*{arg\,min}_{x\in\mathcal{X}} \left\{ \langle \nabla \ell_t(x_t), x\rangle + \sum_{s=1}^{t} D_{\ell_s}(x, x_t) \right\}, \qquad t \geqslant 1,$$

and derive regret guarantees.

Exercice 3 (*Online Newton step*). — We consider an online convex optimization problem where the loss functions admit quadratic lower bounds as follows. At step $t \geqslant 1$,

- the Decision Maker chooses $x_t \in \mathcal{X}$
- Nature chooses a loss function ℓ_t such that there exists $g_t \in \partial \ell_t(x_t)$ and M_t a positive semi-definite matrix of size $d \times d$ such that:

$$\forall x \in \mathcal{X}, \quad \ell_t(x) - \ell_t(x_t) \geqslant \langle g_t, x - x_t\rangle + \frac{1}{2}\langle x - x_t, M_t(x - x_t)\rangle .$$

  ℓ_t, g_t and M_t are revealed.

Let $\lambda > 0$. For all $t \geqslant 1$, denote $A_t = \frac{1}{2}\left(\lambda I_d + \sum_{s=1}^{t-1} M_s\right)$, where I_d is the identity matrix of size $d \times d$.

1) For each of the three iterations defined below, establish an upper bound on the regret

$$\sum_{t=1}^{\mathrm{T}} \left(\ell_t(x_t) - \ell_t(x)\right), \qquad x \in \mathscr{X},\ \mathrm{T} \geqslant 1.$$

Hint. — Use the lemma from the course involved in the analysis of the Vovk–Azoury–Warmuth algorithm.

(i) Let $x_1 \in \mathscr{X}$ and

$$x_{t+1} = \operatorname*{arg\,min}_{x \in \mathscr{X}} \left\| (x_t - \mathrm{A}_t^{-1} g_t) - x \right\|_{\mathrm{A}_t}, \qquad t \geqslant 1.$$

(ii) Let $x_1 \in \mathscr{X}$ and

$$x_{t+1} = \operatorname*{arg\,min}_{x \in \mathscr{X}} \left\{ \langle -\mathrm{A}_t x_t + g_t, x \rangle + \frac{1}{2} x^\top \mathrm{A}_{t+1} x \right\}, \qquad t \geqslant 1.$$

(iii) Let $y_1 \in \mathbb{R}^d$ and

$$x_t = \operatorname*{arg\,min}_{x \in \mathscr{X}} \left\{ \left\langle -y_1 + \sum_{s=1}^{t-1} g_s, x \right\rangle + \frac{1}{2} \langle x, \mathrm{A}_t x \rangle \right\}, \qquad t \geqslant 1.$$

2) Derive the corresponding regret bounds in the special case of *online portfolio optimization* where $\mathscr{X} = \Delta_d$ and where the loss functions are of the form $\ell_t(x) = -\log \langle r_t, x \rangle$ for some $r_t \in (\mathbb{R}_+^*)^d$.