Evaluation
Online learning
links with optimization and games
Université Paris–Saclay

on the Chambolle–Pock method

Consider an objective function $f + \phi$, where $f : \mathbb{R}^d \to \mathbb{R}$ is a convex differentiable function, and $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ a proper, convex, lower semicontinuous function. We assume that the proximal operator associated with ϕ:

$$\mathrm{Prox}_{\phi}(x) = \operatorname*{arg\,min}_{x' \in \mathbb{R}^d} \left\{ \phi(x') + \frac{1}{2} \left\| x' - x \right\|_2^2 \right\}, \qquad x \in \mathbb{R}^d.$$

is easily computable. The *proximal gradient method*, aka *forward-backward splitting* is defined as follows: let $x_0 \in \mathbb{R}^d$ such that $\partial\phi(x_0) \neq \varnothing$, $(\gamma_t)_{t \geqslant 0}$ a positive sequence, and

$$x_{t+1} = \mathrm{Prox}_{\gamma_t \phi}\left(x_t - \gamma_t \nabla f(x_t)\right), \qquad t \geqslant 0.$$

1) Prove that the proximal gradient method is an instance of UMD iterates.

2) Let $L > 0$. In the case where f is L-smooth for $\|\cdot\|_2$, establish for the proximal gradient method a convergence guarantee that extends the classical guarantee for projected gradient descent.

The remaining of this evaluation subject can be considered as bonus. We now turn to Douglas–Rachford splitting. Consider objective function $f+g$ where $f, g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ are simply assumed convex, proper, and lower semicontinuous. A point $x_* \in \mathbb{R}^d$ is a minimizer of $f+g$ if, and only if, $0 \in \partial f(x_*) + \partial g(x_*)$. We assume that such a solution exists. Consider operator

$$F(x) = x + \mathrm{Prox}_g(2 \cdot \mathrm{Prox}_f(x) - x) - \mathrm{Prox}_f(x), \qquad x \in \mathbb{R}^d.$$

The Douglas-Rachford iteration is defined as $x_0 \in \mathbb{R}^d$ and

$$x_{t+1} = F(x_t), \qquad t \geqslant 0.$$

3) Prove that $x_* \in \mathbb{R}^d$ is a fixed point of F if, and only if,

$$0 \in \partial f(\mathrm{Prox}_f(x_*)) + \partial g(\mathrm{Prox}_f(x_*)).$$

4) Prove that F is co-coercive.

5) Derive a guarantee for the Douglas–Rachford iteration.

6) Propose an AdaGrad-based iteration for finding an approximate fixed point of F and derive corresponding guarantees.

7) Conduct a similar approach for the Chambolle–Pock method.